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Aspects linéaires et non-linéaires en géométrie asymptotique des espaces de Banach

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Introduction

1. Résumé de la thèse

Un plongement grossièrement lipschitzien d'un espace métrique M dans un espace métrique N est une application bi-lipschitzienne aux grandes distances (voir le chapitre 1 pour les définitions précises). Dans le papier précurseur [95], M. Ribe a montré que si un espace de Banach X se plonge de façon grossièrement lipschitzienne dans un espace de Banach Y, alors X est finiment crûment représentable dans Y, ce qui signifie que tous les sous-espaces de dimension finie de X sont linéairement isomorphes, avec une distorsion uniforme, aux sous-espaces de Y. En d'autres termes, les propriétés locales des espaces de Banach, qui sont les propriétés isomorphiques de leurs sous-espaces de dimension finie (comme le type, le cotype, la super-réflexivité, $\ldots$) sont préservées par plongements grossièrement lipschitziens. Ce résultat a lancé ce que l'on appelle désormais le programme de Ribe, qui a pour but de trouver des caractérisations purement métriques aux propriétés locales des espaces de Banach. Nous faisons référence à [81] et [5] pour une discussion sur les origines et motivations de ce programme, ainsi qu'une présentation des résultats les plus frappants dans cette direction. Au cours de ces vingt dernières années, la structure asymptotique des espaces de Banach qui, très vaguement parlant, est celle des sous-espaces de codimension finie ou des propriétés des suites ou arbres faiblement nuls, s'est aussi révélée être centrale dans la géométrie non-linéaire des espaces de Banach. Nous renvoyons le lecteur aux travaux fondateurs de N. Kalton ([65] et [63] par exemple) et au panorama [47] ainsi que leurs références. Cependant, malgré l'accumulation d'un certain nombre de résultats de stabilité conséquents, il n'y a pas d'analogue du théorème de rigidité de Ribe dans le cadre de la géométrie asymptotique des espaces de Banach. Chaque nouveau résultat demande un argument ad hoc.
Les parties linéaire et non-linéaire de cette thèse aborderont chacune deux types de questions différents. La première partie linéaire consistera en l'étude fine de quatre propriétés asymptotiques de lissité, notées $\mathsf{T}_p, \mathsf{A}_p, \mathsf{N}_p$ et P_p, en mettant l'accent sur des caractérisations par renormage et le problème des trois-espaces. Ces caractérisations par renormage et le principe de Gorelik nous emmèneront vers des résultats non-linéaires en nous permettant d'exhiber deux nouvelles propriétés qui sont stables par équivalences grossièrement lipschitziennes. Notre deuxième thème non-linéaire sera l'étude de propriétés de concentration pour des applications lipschitziennes définies sur une famille de graphes, qui peuvent empêcher le plongement grossièrement lipschitzien d'un espace de Banach dans un autre. Enfin, la deuxième partie linéaire consistera à donner un analogue asymptotique d'un célèbre résultat de Pisier impliquant plusieurs notions de types et la B-convexité.

Décrivons maintenant en détails la structure de cette thèse.

Chapitre 1 : Applications non-linéaires et propriétés asymptotiques des espaces de Banach.

Commençons par mentionner que la deuxième section de ce chapitre est tirée d'un travail conjoint avec R. Causey et G. Lancien (voir [30]).

Dans ce premier chapitre, nous définissons les applications non-linéaires qui apparaissent dans cette thèse et, après quelques rappels sur l'indice de Szlenk, nous décrivons en détails quatre propriétés différentes en lien avec la lissité asymptotique uniforme des espaces de Banach, que nous noterons $\mathsf{T}_p, \mathsf{A}_p, \mathsf{N}_p$, et P_p. Nous commencerons tout d'abord par donner leurs définitions en termes de jeux à deux joueurs sur un espace de Banach X.

Ensuite, nous donnons les caractérisations principales de ces propriétés qui font intervenir des majorations de type ℓ_p pour des arbres faiblement nuls, l'existence de normes équivalentes asymptotiquement uniformément lisses avec de bonnes estimations quantitatives et, dualement, la comportement de l'indice de Szlenk.

Nous supposons, pour cette introduction, que ce que nous voulons dire par arbres faiblement nuls est compréhensible. Nous pouvons alors énoncer le résultat principal du chapitre 1. Il donne plusieurs caractérisations de la propriété A_p. La caractérisation avec le renormage est complètement nouvelle et sera cruciale pour les résultats de stabilité non-linéaire.

Théorème (avec R. Causey et G. Lancien)**.** *Soient $1 < p < \infty$ et q l'exposant conjugué de p. Soit X un espace de Banach. Les assertions suivantes sont équivalentes*

(i) $X \in \mathsf{A}_p$.

(ii) Il existe une constante $c > 0$ telle que pour toute base de voisinages faibles D de 0 dans X, tout $n \in \mathbb{N}$, et tout arbre faiblement nul $(x_t)_{t\in D^{\leqslant n}}$ dans la boule unité de X, il existe $t \in D^n$ tel que

$$\forall a = (a_i)_{i=1}^n \in \ell_p^n, \quad \Big\| \sum_{i=1}^n a_i x_{t|_i} \Big\| \leq c\|a\|_p.$$

(iii) Il existe une constante $M \geq 1$ et une constante $C > 0$ telles que pour tout $\tau \in (0,1]$, il existe une norme $|\ \ |$ sur X vérifiant $M^{-1}\|x\|_X \leq |x| \leq M\|x\|_X$ pour tout $x \in X$ et

$$\forall \sigma \geq \tau, \quad \overline{\rho}_{|\ |}(\sigma) \leq C\sigma^p.$$

(iv) X a un indice de Szlenk q-sommable.

Rappelons qu'une propriété (P) concernant les espaces de Banach est séparablement déterminée si un espace de Banach X a (P) si et seulement si tous ses sous-espaces séparables ont (P). Nous terminerons ce chapitre par une preuve courte et unifiée du fait que les quatre propriétés introduites sont séparablement déterminées.

Chapitre 2 : Graphes de Hamming et propriétés de concentration dans des espaces de Banach non-quasi-réflexifs.

En 2008, afin de montrer que $L_p(0,1)$ n'est pas uniformément homéomorphe à $\ell_p \oplus \ell_2$ pour $p \in (1,\infty)\setminus\{2\}$, Kalton et Randrianarivony [67] ont introduit une nouvelle technique s'appuyant sur une certaine famille de graphes et des idées de lissité asymptotique. Plus spécifiquement, ils ont introduit une propriété de concentration pour les applications lipschitziennes définies sur les graphes de Hamming à valeurs dans un espace de Banach réflexif asymptotiquement uniformément lisse (AUS) X, qui empêche tout plongement grossièrement lipschitzien de certains espaces dans X.

Leur résultat fut utilisé par Kalton lui-même pour en déduire des informations sur les modèles étalés d'un espace qui se plonge de façon grossièrement lipschitzienne dans un espace réflexif AUS (cf [65]), et a été étendu au cadre quasi-réflexif par Lancien et Raja [74], qui ont introduit une propriété de concentration plus faible. Peu après, Causey [28] a montré que cette même propriété de concentration plus faible se retrouve également chez les espaces quasi-réflexifs admettant des majorations de type ℓ_p pour les arbres (plus précisément, chez les espaces quasi-réflexifs qui appartiennent à la classe N_p introduite dans le chapitre 1).

L'objectif de ce chapitre est de démarrer une étude générale de ces propriétés de concentration, ainsi que de nouvelles. En particulier, nous nous poserons la question de leur stabilité par des sommes indexées par certains espaces de Banach. Ceci nous permettra d'obtenir des exemples non-quasi-réflexifs.
Notre résultat principal sera le suivant.

Théorème. *Soient $p \in]1, \infty[$, $\lambda > 0$ et $(X_n)_{n\in\mathbb{N}}$ une suite d'espaces de Banach avec la propriété λ-HIC$_{p,d}$.*
Soit E un espace de Banach réflexif avec une base normalisée 1-inconditionnelle et p-convexe $(e_n)_{n\in\mathbb{N}}$ (voir la fin de l'introduction pour la définition), avec constante de convexité 1.
Alors $X = \left(\sum_{n\in\mathbb{N}} X_n\right)_E$ a la propriété $(\lambda + 2 + \varepsilon)$-HIC$_{p,d}$ pour tout $\varepsilon > 0$.

La propriété λ-HIC$_{p,d}$ est un raffinement de la propriété λ-HIC$_p$, d'abord considérée par Lancien et Raja : un espace X a la propriété λ-HIC$_p$ si, pour toute fonction lipschitzienne $f : ([\mathbb{N}]^k, d_{\mathbb{H}}) \to X$, il existe $\overline{n}, \overline{m}$ en position entrelacée tels que $\|f(\overline{n}) - f(\overline{m})\| \leq \lambda k^{1/p}\mathrm{Lip}(f)$.

Par conséquent, une somme ℓ_q d'un espace de Banach quasi-réflexif (voir la fin de l'introduction pour la définition) satisfaisant des majorations de type ℓ_p pour les arbres, $1 < q, p < \infty$, ne peut contenir de copie équi-lipschitzienne des graphes de Hamming. C'est une généralisation du résultat de Causey mentionné précédemment (cf [28]), qui le prouve pour les espaces de Banach quasi-réflexifs satisfaisant des majorations de type ℓ_p pour les arbres. C'est le premier résultat de ce type pour les espaces de Banach non-quasi-réflexifs.

Pour le montrer, on introduit les notions et la terminologie utilisées par la suite dans la première section tandis que la section 2 est dédiée à la preuve elle-même.

Récemment, Baudier, Lancien, Motakis and Schlumprecht [11] ont prouvé que tout espace de Banach quasi-reflexif asymptotiquement c_0 (voir la section 3 pour la définition d'asymptotiquement c_0) a la propriété HIC$_\infty$. Même si nous ne savons toujours pas si un espace qui a cette propriété est nécessairement quasi-réflexif, nous prouvons dans la dernière section de ce chapitre qu'un espace avec la propriété HIC$_\infty$ est nécessairement asymptotiquement c_0. En particulier, l'espace $T^*(T^*)$, où T^* est l'espace de Banach original construit par Tsirelson dans [98], ne peut avoir cette propriété de concentration.
On donne également un exemple dans le cadre non-quasi-réflexif d'un espace dual, séparable et asymptotiquement c_0 qui ne contient pas de copie lipschitzienne de ℓ_1 ou c_0 et

qui n'a aucune des propriétés de concentration introduites dans ce chapitre. Cet exemple s'appuie sur une généralisation de la construction des espaces de Lindenstrauss, que l'on doit à Schlumprecht.

Chapitre 3 : Propriétés des trois-espaces et stabilités non-linéaires.

Ce chapitre est en deux parties et est basé sur un travail conjoint avec R. Causey et G. Lancien [30]. La première est dédiée à l'étude du problème des trois-espaces pour les propriétés $\mathsf{T}_p, \mathsf{A}_p, \mathsf{N}_p$ et P_p, introduites dans le premier chapitre. La deuxième est quant à elle dédiée à des résultats de stabilité non-linéaire.

Rappelons qu'une propriété (P) d'espaces de Banach est une *Propriété des trois-espaces* (ou 3SP en abrégé) si elle passe aux quotients et sous-espaces et qu'un espace de Banach X a (P) dès lors qu'il admet un sous-espace Y tel que Y et X/Y ont tous les deux (P). Les propriétés $\mathsf{T}_p, \mathsf{A}_p, \mathsf{N}_p$ et P_p passent assez simplement aux sous-espaces, quotients ou espaces isomorphes et il a été prouvé dans [29] que P_p est une 3SP. Pour commencer, nous profitons de ce chapitre pour fournir un argument plus direct pour le montrer. Ensuite, avec un seul exemple, nous montrons

Théorème. *Soit $p \in (1, \infty)$. Alors $\mathsf{T}_p, \mathsf{A}_p,$ et N_p ne sont pas des propriétés des trois-espaces.*

Enfin, et ceci sera notre résultat principal concernant les propriétés des trois-espaces, nous montrons

Théorème. *Admettre un renormage asymptotiquement uniformément plat (propriété T_∞) et avoir un indice de Szlenk sommable (propriété A_∞) sont des propriétés des trois-espaces.*

Un réseau dans un espace métrique (M, d) est un sous-ensemble $\mathcal{M}$ de M tel qu'il existe $0 < a < b$ de sorte que pour tous $z \neq z'$ dans $\mathcal{M}$, $d(z, z') \geq a$ et pour tout x dans M, $d(x, \mathcal{M}) < b$. Pour que ce soit plus simple dans cette introduction, utilisons le fait que deux espaces de Banach X et Y de dimension infinie sont grossièrement Lipschitz équivalents si et seulement si on peut trouver deux réseaux de X et Y qui sont Lipschitz équivalents. La définition précise d'équivalence grossièrement lipschitzienne sera donnée dans le chapitre 1 et peut-être décrite approximativement comme une équivalence lipschitzienne aux grandes distances. Dans [45] et [46], il est prouvé que T_p est stable par équivalences lipschitziennes, que P_p est stable par équivalences grossièrement lipschitziennes et que $\mathsf{A}_\infty = \mathsf{N}_\infty$ est stable par équivalences grossièrement lipschitziennes. Dans [64], N. Kalton a prouvé que pour $1 < p < \infty$, la classe T_p n'est pas stable par équivalences grossièrement lipschitziennes. Grâce à nos théorèmes de renormage du chapitre 1, on peut presque clôturer cet ensemble de résultats.

Théorème. *Soit $p \in (1, \infty)$. Les classes A_p et N_p sont stables par équivalences grossièrement lipschitziennes.*

Indiquons que, pour N_p, nous le déduisons de résultats déjà existants mais que cela n'avait pas été remarqué tandis que, pour A_p, cela repose sur notre nouvelle caractérisation de renormage.

Nous concluons ce chapitre en citant quelques exemples connus d'espaces ayant T_∞ ou A_∞ et quelques questions.

Chapitre 4 : Quelques types asymptotiques.

Ce chapitre est basé sur un travail mené avec Florent Baudier.

Un des premiers résultats du programme de Ribe est la caractérisation purement métrique de Bourgain-Milman-Wolfson [17] des espaces de Banach ayant un type linéaire non trivial. Cette caractérisation de Bourgain-Milman-Wolfson repose sur une modification de la notion (non-linéaire) de type d'Enflo, que l'on appelle désormais type BMW. Les types d'Enflo et BMW sont intimement connectés à la géométrie de la suite des cubes de Hamming $\{H_n\}_{n\in\mathbb{N}}$. Il est immédiat de vérifier qu'un espace métrique avec un type d'Enflo ou BMW non trivial ne contient pas de copie bi-lipschitzienne des cubes de Hamming avec une distorsion bornée uniforme (et on peut le faire de façon quantitative). Il a été montré dans [17] que la réciproque est vraie pour le type BMW. La question de la réciproque pour le type d'Enflo reste un problème ouvert important. Puisque le cube de Hamming n-dimensionnel H_n se plonge naturellement isométriquement dans ℓ_1^n, il s'ensuit que tout espace de Banach qui contient uniformément (au sens linéaire) $\{\ell_1^n\}_{n\in\mathbb{N}}$ contient uniformément (au sens bi-lipschitzien) les cubes de Hamming. On sait grâce à un célèbre résultat de Pisier [91] que les espaces de Banach contenant uniformément des copies des ℓ_1^n sont exactement ceux qui ont un type linéaire trivial. Par conséquent, si un espace de Banach ne contient pas de copie bi-lipschitzienne des cubes de Hamming avec une distorsion bornée uniforme, alors il doit avoir un type linéaire non trivial. Bourgain, Milman et Wolfson ont montré qu'un espace de Banach dont le type linéaire vaut $p \in (1,2)$ a nécessairement type BMW $p-\varepsilon$, pour tout $\varepsilon \in (0, p-1)$. La preuve de Pisier du résultat analogue pour la notion de type d'Enflo est basée sur une inégalité fondamentale, désormais appelée inégalité de Pisier. Partant de cette discussion, on a équivalence entre les assertions suivantes pour un espace de Banach X.

1. X a un type linéaire non trivial.
2. X ne contient pas uniformément les ℓ_1^n, $n \in \mathbb{N}$.
3. X ne contient pas de copies bi-lipschitzienne des cubes de Hamming avec une distorsion bornée uniforme.
4. X a un type BMW non trivial.
5. X a un type d'Enflo non trivial.

Comme il a été mentionné plus haut, l'équivalence entre (3) et (4) est valable pour n'importe quel espace métrique et est un analogue purement métrique de l'équivalence entre (1) et (2), qui n'a de sens que pour les espaces de Banach. L'équivalence entre (1) et (4) est un exemple de caractérisation métrique d'une propriété locale via des inégalités de type Poincaré, tandis que l'équivalence entre (1) et (3) s'exprime grâce à l'exclusion d'une suite de graphes finis. C'est typique dans le programme de Ribe. La relation quantitative entre type linéaire et type d'Enflo a été complètement élucidée par von Handel, Ivanishvili, et Volberg [54]. Le problème d'Enflo demandait si un espace de Banach avec type linéaire p devait avoir un type d'Enflo égal à p (la réciproque étant clairement vraie). La réponse à ce problème vieux d'alors 40 ans s'est révélée être positive dans [54], et peut désormais être utilisée afin de fournir une autre preuve à certaines des équivalences ci-dessus.

La recherche de caractérisations métriques à des propriétés asymptotiques a débuté avec [8]. Depuis la publication de [8], l'exploration d'une facette asymptotique du programme de Ribe a mené à des résultats significatifs qui mettent en évidence le lien entre propriétés asymptotiques des espaces de Banach et géométrie de suites de graphes dénombrables infinis. En particulier, des analogues asymptotiques de deux caractérisations métriques influentes de la super-réflexivité ont été découverts dans [8] et [7]. Les arbres à branchements dénombrables [8] et les diamants à branchements dénombrables [7] jouent le rôle des arbres dyadiques dans [16] et des graphes diamants dyadiques dans [59], respectivement. Il est intéressant de noter que le cadre asymptotique offre une plus grande variété de phénomènes géométriques potentiels, et que certaines caractérisations métriques utiles de propriétés asymptotiques semblent ne pas avoir d'analogue dans le programme de Ribe (voir [12], [11] pour un exemple marquant).

Formuler un analogue asymptotique du problème d'Enflo, ou de la caractérisation BMW, nous échappe encore et est un des problèmes fondamentaux du programme de Ribe asymptotique. L'une des raisons étayant la difficulté de cette tâche est qu'il semble ne pas y avoir de façon canonique de définir ce qu'est le type linéaire asymptotique d'un espace de Banach. Une approche en termes de structure asymptotique à la Maurey-Milman-Tomczak-Jaergerman a été entreprise dans [29], où un analogue asymptotique de l'équivalence linéaire entre (1) et (2) a été prouvé. Les graphes de Hamming, analogues asymptotiques des cubes de Hamming, se sont déjà révélés être étroitement connectés à la géométrie asymptotique des espaces de Banach, et leur géométrie imite en un sens celle de leurs analogues finis. Cependant, la géométrie des graphes de Hamming est plus naturellement liée à la notion de modèles asymptotiques qu'à celle de structure asymptotique, comme expliqué dans [11].

Formuler un analogue asymptotique cohérent et pertinent au problème d'Enflo passe par une meilleure appréhension des notions de "type linéaire asymptotique". Nous voulons pointer du doigt le fait que la classe des espaces de Banach super-réflexifs peut être caractérisée géométriquement en termes de renormages uniformément lisses et/ou convexes tandis que l'analogue asymptotique de ces renormages mène à des classes d'espaces de Banach distinctes qui admettent des caractérisations métriques différentes. Avoir un type linéaire non trivial possède de nombreuses caractérisations équivalentes, par exemple en termes de B-convexité, d'infratype ou de type stable. Par conséquent, il se pourrait que les analogues asymptotiques de ces notions mènent à différentes classes de Banach dans le monde asymptotique, ce qui multiplie alors les candidats pour une éventuellement formulation du problème d'Enflo asymptotique.

Dans ce chapitre, nous conduisons une étude systématique de ces notions depuis une perspective asymptotique. Nous proposons des analogues asymptotiques naturels aux notions de B-convexité, d'infratype et de type stable, et nous montrons que les caractérisations équivalentes en termes de type non trivial dans le cadre local restent vraies dans le cadre asymptotique. Une fois les notions asymptotiques correctement définies, certaines preuves de la théorie locale pourront être transférées dans le cadre asymptotique de façon parfois immédiate mais certains arguments locaux reposant sur la sous-multiplicativité de constantes associées aux types nécessiteront des modifications non triviales.

Annexe A : Distance entre c_0 and certains $C(K)$.

Dans [21], Cambern prouve que la distance de Banach-Masur entre c, l'espace des suites convergentes, et c_0 vaut 3. Plus de quarante ans plus tard, Candido et Galego ont étendu ce résultat en montrant que $d(c_0, C([0, \omega^n])) = 2n + 1$ pour tout $1 \leq n < \omega$ (voir [22]), à l'aide d'arguments venant majoritairement de la théorie de la mesure. Le but de cette annexe est de donner une nouvelle preuve plus courte de l'inégalité $d(c_0, C([0, \omega^n])) \geq 2n + 1$ grâce à des arguments issus de la géométrie asymptotique des espaces de Banach.

Annexe B : Quelques indices non-linéaires.

Dans [10], Baudier, Lancien, Motakis et Schlumprecht ont exhibé une famille d'espaces $\mathscr{C}$, indexée par ω_1 et dont aucun de ses membres n'est Lipschitz universel, qu'ils appellent la famille des espaces métriques lisses de Schreier à valeurs rationnelles, qui contient assez d'informations sur la structure de c_0 pour qu'un espace métrique soit Lipschitz universel pour la classe de tous les espaces métriques séparables s'il l'est pour $\mathscr{C}$ (on rappelle que, d'après le théorème d'Aharoni [1], c_0 est Lipschitz universel pour tous les espaces métriques séparables). Pour ce faire, ils ont généralisé l'indice ordinal de Bourgain "mesurant" le degré de présence d'une suite basique donnée dans un espace de Banach, en utilisant des vignes à la places des arbres.
Après quelques rappels, nous montrerons avec des arguments similaires que des familles semblables peuvent témoigner de la présence lipschitzienne de ℓ_p, $1 \leq p < \infty$. Nous pourrons en déduire qu'un espace de Banach contenant une copie bi-lipschitzienne de tout espace réflexif asymptotiquement ℓ_1 contient une copie lipschitzienne de ℓ_1. Une fois ce résultat prouvé, nous verrons qu'il peut être fait de même pour les plongements grossièrement lipschitziens et qu'un espace de Banach contient une copie grossièrement lipschitzienne de c_0 s'il contient une copie grossièrement lipschitzienne de tout espace réflexif et asymptotiquement c_0. Pour finir, nous prouverons les équivalences suivantes.

Proposition. *Soient X un espace de Banach et $1 \leq p < \infty$. Les assertions suivantes sont équivalentes :*

(i) $I_{\ell_p}^{Lip}(X) > \omega$;

(ii) $I_{\ell_p}^{cL}(X) > \omega$;

(iii) *ℓ_p est finiment représentable dans X.*

De même, $I_{c_0}^{Lip}(X) > \omega$ si et seulement si $I_{c_0}^{cL}(X) > \omega$ si et seulement si c_0 est finiment représentable dans X.

où $\mathrm{I}_{\ell_p}^{\mathrm{Lip}}(X)$ (respectivement $\mathrm{I}_{\ell_p}^{\mathrm{cL}}(X)$) est l'indice ordinal permettant de mesurer le degré de présence lipschitzienne (respectivement grossièrement lipschitzienne) de ℓ_p dans X.

2. Summary of the thesis

A coarse Lipschitz embedding from a metric space M into a metric space N is a map which is bi-Lipschitz for large enough distances (see precise definitions in Chapter 1). In the seminal paper [95], M. Ribe proved that if a Banach space X coarse Lipschitz embeds into a Banach space Y, then X is finitely crudely representable into Y, which means that all finite-dimensional subspaces of X are linearly isomorphic, with a uniform distortion, to subspaces of Y. In other words, the local properties of Banach spaces, that are isomorphic properties of their finite dimensional subspaces (such as type, cotype, super-reflexivity,...) are preserved under coarse Lipschitz embeddings. This initiated what is now called the Ribe program, which aims at finding purely metric characterizations of local properties of Banach spaces. We refer to [81] and [5] for a discussion of the origins and motivations of this program, and for a presentation of the most striking results in this direction. In the last twenty years, the asymptotic structure of Banach spaces, which, very vaguely speaking, deals with its of their finite-codimensional subspaces or with the properties of weakly null sequences and trees, also proved to be central in the non-linear geometry of Banach spaces. We refer the reader to the seminal works of N. Kalton ([65] and [63] for instance) and to the survey [47] and references therein. However, despite the accumulation of quite a few important stability results, there is no general analogue of Ribe's rigidity theorem in the setting of the asymptotic geometry of Banach spaces. Every new result requires an ad hoc argument.
Each linear and non-linear part of this work will deal with two different kinds of questions. The first linear part will consist in a sharp study of four asymptotic smoothness properties denoted $\mathsf{T}_p, \mathsf{A}_p, \mathsf{N}_p$, and P_p, with an emphasis on renorming characterizations and the three-space problem. Those renorming characterizations and the Gorelik principle will drive us towards non-linear results by allowing us to exhibit two new properties that are stable under coarse Lipschitz equivalences. Our second non-linear theme will be the study of some concentration properties for Lipschitz maps defined on a family of graphs that can prevent the coarse Lipschitz embedding of a Banach space into another. Finally, the second linear part will consist in giving an asymptotic analogue of a famous local result by Pisier involving several notions of types and B-convexity.

Let us now describe the structure of this thesis.

Chapter 1: Non-linear maps and asymptotic properties of Banach spaces.

First, let us mention that the second section of this chapter is based on a joint work with R. Causey and G. Lancien (see [30]).

In this first chapter, we define the non-linear maps that play a role in this thesis and, after some reminders on the Szlenk index, we describe in details four different properties dealing with the asymptotic uniform smoothness of Banach spaces, that we shall denote $\mathsf{T}_p, \mathsf{A}_p, \mathsf{N}_p$, and P_p. First, we start by giving their definitions in terms of two-players games on a Banach space X.

Then, we give the main characterizations of these properties, which involve, upper ℓ_p estimates for weakly null trees, the existence of quantitatively good equivalent asymptotically uniformly smooth norms and, dualy, the behaviour of the Szlenk index.

We assume, for this introduction, that what we mean by weakly null trees is understandable. We can now state our main new result from Chapter 1. It describes various characterizations of our property A_p. The renorming characterization is completely new and will be crucial for our non-linear stability results.

Theorem A (with R. Causey and G. Lancien)**.** *Fix $1 < p < \infty$ and let q be conjugate to p. Let X be a Banach space. The following are equivalent*

(i) $X \in \mathsf{A}_p$.

(ii) There exists a constant $c > 0$ such that for any weak neighborhood base D at 0 in X, any $n \in \mathbb{N}$, and any weakly null tree $(x_t)_{t\in D^{\leqslant n}}$ in the unit ball of X, there exists $t \in D^n$ such that

$$\forall a = (a_i)_{i=1}^n \in \ell_p^n, \quad \Big\| \sum_{i=1}^n a_i x_{t|_i} \Big\| \leq c\|a\|_p.$$

(iii) There exists a constant $M \geq 1$ and a constant $C > 0$, such that for any $\tau \in (0,1]$ there exists a norm $|\ |$ on X satisfying $M^{-1}\|x\|_X \leq |x| \leq M\|x\|_X$ for all $x \in X$ and

$$\forall \sigma \geq \tau, \quad \overline{\rho}_{|\ |}(\sigma) \leq C\sigma^p.$$

(iv) X has q-summable Szlenk index.

Let us recall that a property (P) of Banach spaces is separably determined if a Banach space X has (P) if and only if all its separable subspaces have (P). We finish this chapter by providing a short and unified proof of the fact that the four properties introduced here are separably determined.

Chapter 2: Hamming graphs and concentration properties in non-quasi-reflexive Banach spaces.

In 2008, in order to show that $L_p(0,1)$ is not uniformly homeomorphic to $\ell_p \oplus \ell_2$ for $p \in (1,\infty) \setminus \{2\}$, Kalton and Randrianarivony [67] introduced a new technique based on a certain class of graphs and asymptotic smoothness ideas. To be more specific, they introduced a concentration property for Lipschitz maps defined on Hamming graphs into a reflexive asymptotically uniformly smooth (AUS) Banach space X, that prevents coarse Lipschitz embeddings of certain other spaces into X. Their result was used by Kalton himself to deduce some information about the spreading models of a space that coarse Lipschitz embeds into a reflexive AUS space (see [65]), and was later extended to the quasi-reflexive setting by Lancien and Raja [74], who introduced a weaker concentration property. Soon after, Causey [28] proved that quasi-reflexive spaces admitting so-called upper ℓ_p tree estimates also have this same weaker concentration property (more precisely, quasi reflexive spaces belonging to the class N_p introduced in Chapter 1).

The purpose of this chapter is to start a general study of these concentration properties, together with new ones. In particular, we will address the question of their stability under sums of Banach spaces. This will allow us to get non-quasi-reflexive examples.
Our main result will be the following.

Theorem B. *Let $p \in (1,\infty)$, $\lambda > 0$, $(X_n)_{n\in\mathbb{N}}$ a sequence of Banach spaces with property λ-HIC$_{p,d}$.*
Let E be a reflexive Banach space with a normalized 1-unconditional p-convex basis $(e_n)_{n\in\mathbb{N}}$ with convexity constant 1.
Then $X = \left(\sum_{n\in\mathbb{N}} X_n\right)_E$ has property $(\lambda+2+\varepsilon)$-HIC$_{p,d}$ for every $\varepsilon > 0$.

Property λ-HIC$_{p,d}$ is a refinement of the property λ-HIC$_p$ first considered by Lancien and Raja: a space X has property λ-HIC$_p$ if for any Lipschitz function $f : ([\mathbb{N}]^k, d_{\mathbb{H}}) \to X$, there exist $\overline{n}, \overline{m}$ in interlacing position such that $\|f(\overline{n}) - f(\overline{m})\| \le \lambda k^{1/p}\mathrm{Lip}(f)$.

As a consequence, we get that an ℓ_q-sum of a quasi-reflexive Banach space satisfying upper ℓ_p tree estimates, $1 < q, p < \infty$, cannot equi-Lipschitz contain the Hamming graphs. This is a generalization of the result mentioned above by Causey (see [28]), who proved it for quasi-reflexive Banach spaces satisfying upper ℓ_p tree estimates. This is the first result of this type for non-quasi-reflexive Banach spaces.

In order to show this result, we introduce the notions and the terminology we will use later in the first section while Section 2 is dedicated to the proof itself.

Recently, Baudier, Lancien, Motakis and Schlumprecht [11] proved that any quasi-reflexive asymptotic-c_0 Banach space (see Section 3 for the definition of asymptotic-c_0) has property HIC$_\infty$. Even though we don't know if a Banach space with this property is quasi-reflexive, we prove in the last section of this chapter that property HIC$_\infty$ implies asymptotic-c_0. In particular, the space $T^*(T^*)$, where T^* is the original Banach space constructed by Tsirelson in [98], cannot have this concentration property.
We also give an example in the non-quasi-reflexive setting of a separable dual asymptotic-c_0 space that does not Lipschitz contain ℓ_1 nor c_0 and without any of the concentration properties introduced in this chapter. This example is based on a generalization of the construction of Lindenstrauss spaces, that is due to Schlumprecht.

Chapter 3: Three-space properties and non-linear stabilities.

This chapter is in two parts and is based on a joint work with Ryan Causey and Gilles Lancien [30]. The first one is devoted to the study of the three-space problem for $\mathsf{T}_p, \mathsf{A}_p, \mathsf{N}_p$ and P_p, introduced in the first chapter. The second one is dedicated to non-linear stability results.

Let us recall that a property (P) of Banach spaces is a *Three-Space Property* (3SP in short) if it passes to quotients and subspaces and a Banach space X has (P) whenever it admits a subspace Y such that Y and X/Y have (P). The properties $\mathsf{T}_p, \mathsf{A}_p, \mathsf{N}_p$ and P_p pass quite simply to subspaces, quotients or isomorphs and it was proved in [29] that P_p is a 3SP. First, we take the opportunity of this chapter to provide a more direct argument for this. Then, with a single example, we show

Theorem C. *Let $p \in (1,\infty)$. Then $\mathsf{T}_p, \mathsf{A}_p$, and N_p are not three-space properties.*

Finally, and this is the main result about three-space properties, we show

Theorem D. *Asymptotic uniform flattenability (property T_∞) and summable Szlenk index (property A_∞) are three-space properties.*

A net in a metric space (M, d) is a subset $\mathcal{M}$ of M such that there exist $0 < a < b$ so that for every $z \neq z'$ in $\mathcal{M}$, $d(z, z') \geq a$ and for every x in M, $d(x, \mathcal{M}) < b$. Let us use, for the simplicity of this introduction, that two infinite dimensional Banach spaces X and Y are coarse Lipschitz equivalent if and only if there exist two nets in X and Y that are Lipschitz equivalent. The precise definition of a coarse Lipschitz equivalence is given in Chapter 1 and can be roughly described as a Lipschitz equivalence at large distances. In [45] and [46], it is proved that T_p is stable under Lipschitz equivalences, P_p is stable under coarse Lipschitz equivalences and $\mathsf{A}_\infty = \mathsf{N}_\infty$ is stable under coarse Lipschitz equivalences. In [64], N. Kalton proved that for $1 < p < \infty$, the class T_p is not stable under coarse Lipschitz equivalences. Thanks to our renorming theorems from Chapter 1, we can almost complete this set of results.

Theorem E. *Let $p \in (1, \infty)$. Then, the class A_p and the class N_p are stable under coarse Lipschitz equivalences.*

Let us point that for N_p, this is deduced from already existing results, but was unnoticed, while for A_p it relies on our new renorming characterization.

We conclude this chapter by gathering a few known examples of T_∞ or A_∞ spaces and related questions.

Chapter 4: Some asymptotic types.

This chapter is based on a joint work with Florent Baudier.

One of the earliest results in the Ribe program is the Bourgain-Milman-Wolfson purely metric characterization of Banach spaces with non-trivial linear type. The Bourgain-Milman-Wolfson [17] characterization is based on a modification of the (non-linear) notion of Enflo type, which has come to be referred to as BMW type. Enflo and BMW types are intimately connected to the geometry of the sequence of Hamming cubes $\{H_n\}_{n\in\mathbb{N}}$. It is immediate to verify that any metric space with non-trivial Enflo or BMW type does not contain bi-Lipschitz copies of the Hamming cubes with uniformly bounded distortion (and this can be made quantitative). It was shown in [17] that the converse holds for BMW type. Whether the converse holds for Enflo type is an important open problem. Since the n-dimensional Hamming cube H_n naturally isometrically embeds into ℓ_1^n, it follows that every Banach space that contains $\{\ell_1^n\}_{n\in\mathbb{N}}$ uniformly (in the linear sense) will contain the Hamming cubes uniformly (in the bi-Lipschitz sense). It is a famous theorem of Pisier [91] that Banach spaces containing uniformly isomorphic copies of the ℓ_1^n's are exactly the Banach spaces with trivial linear type. Therefore, if a Banach space does not contain bi-Lipschitz copies of the Hamming cubes with uniformly bounded distortion, then it must have non-trivial linear type. Bourgain, Milman, and Wolfson showed that if a Banach space has linear type $p \in (1, 2)$ then necessarily it has BMW type $p - \varepsilon$, for all $\varepsilon \in (0, p-1)$. Pisier's proof of the analogous result for the notion of Enflo type is based on a fundamental inequality, nowadays called Pisier's inequality. Based on the discussion above, we have that for any Banach space X, the following assertions are equivalent.

1. X has non-trivial linear type.

2. X does not contain ℓ_1^n's uniformly.

3. X does not contain bi-Lipschitz copies of the Hamming cubes with uniformly bounded distortion.

4. X has non-trivial BMW type.

5. X has non-trivial Enflo type.

As mentioned above, the equivalence between (3) and (4) holds for arbitrary metric spaces and is a purely metric analogue of the equivalence between (1) and (2), which only makes sense for Banach spaces. The equivalence between (1) and (4), for instance, is an example of a metric characterization of a local property in terms of Poincaré-type inequalities, while the equivalence between (1) and (3) is expressed in terms of preclusion of a sequence of finite graphs. These are typical in the Ribe progam. The quantitative relationship between linear type and Enflo type has been completely elucidated by von Handel, Ivanishvili, and Volberg [54]. Enflo's Problem asked whether a Banach space with linear type p must have Enflo type p (that the converse holds is straightfoward). This 40-year old open problem at the time was solved positively in [54], and can now be used to provide alternate proofs of some of the equivalences above.

The study of metric characterizations of asymptotic properties was initiated in [8]. Since the appearance of [8], the investigation of the asymptotic facet of the Ribe program has led to significant results emphasizing the link between asymptotic properties of Banach spaces and the geometry of sequences of countably infinite graphs. In particular, asymptotic analogues of two influential metric characterizations of super-reflexivity were discovered in [8] and [7]. The countably branching trees [8] and the countably branching diamonds [7] play the role of the binary trees in [16], and the binary diamond graphs in [59], respectively. It is worth pointing out that the asymptotic setting offers a finer grain regarding the variety of the potential geometric phenomena that can occur, and there are useful metric characterizations of asymptotic properties which do not seem to have analogues in the local Ribe progam (see [12], [11] for a striking example).

Formulating an asymptotic analogue of Enflo's Problem, or of the BMW characterization, has remained elusive and is one of the fundamental problems in the asymptotic facet of the Ribe program. One of the reasons underpining the difficulty of this task, stands from the fact that there does not seem to be a canonical way to define what is the asymptotic linear type of a Banach space. One approach in terms of Maurey-Milman-Tomczak-Jaergerman asymptotic structure was undertaken in [29] where an asymptotic analogue of the linear equivalence of (1) and (2) was proved. The Hamming graphs, which are countably infinite graphs defined in a similar way as the Hamming cubes, have already shown to be tightly connected to the asymptotic geometry of Banach spaces, and their geometry mimics to some extent the geometry of their finite counterparts. However, the geometry of the Hamming graphs is most naturally connected with the notion of asymptotic models rather than to the notion of asymptotic structure as explained in [11].

In order to formulate a viable and useful asymptotic analogue of Enflo's Problem, one first needs to further our understanding of potential notions of "asymptotic linear type". We want to point out that the class of super-reflexive Banach spaces can be geometrically characterized in terms of equivalent uniformly smooth and/or convex renormings, but the asymptotic analogues of these renormings lead to distinct classes of Banach spaces which admit distinct metric characterizations. Having non-trivial linear type has numerous equivalent characterizations, e.g. in terms of B-convexity, infratype, or stable type. Thus, it could be that considering asymptotic analogues of these notions leads to distinct classes in the asymptotic world, thereby multiplying the candidates for potential formulations of an asymptotic Enflo's Problem.

In this chapter, we conduct a rather systematic study of these notions from the asymptotic perspective. We propose natural asymptotic analogues of B-convexity, infratype, and stable type, and show that all the equivalent characterizations of non-trivial type in terms of the local notions remain equivalent characterizations for their asymptotic counterparts. Some of the local proofs can be transferred somewhat straighforwardly to the asymptotic setting once the asymptotic notions have been adequately defined. However, some of the local arguments relying on the submultiplicativity of various type constants need non-trivial modifications.

Appendix A: Distance between c_0 and some $C(K)$.

In [21], Cambern proves that the value of the Banach-Mazur distance between c, the space of convergent sequences, and c_0 is 3. More than forty years later, Candido and Galego extended this result by showing that $d(c_0, C([0, \omega^n])) = 2n+1$ for every $1 \leq n < \omega$ (see [22]), with arguments mainly coming from measure theory. The purpose of this appendix is to give a new and shorter proof of the inequality $d(c_0, C([0, \omega^n])) \geq 2n + 1$ with arguments coming from asymptotic geometry of Banach spaces.

Appendix B: Some non-linear indices.

In [10], Baudier, Lancien, Motakis and Schlumprecht exhibited an uncountable collection $\mathscr{C}$, which they refer to as the collection of rational-valued smooth Schreier metric spaces, that captures enough structure of c_0 so that a metric space is Lipschitz universal for the class of all separable metric spaces if it is universal for $\mathscr{C}$ (we recall that, by Aharoni's theorem [1], c_0 is Lipschitz universal for all separable metric spaces). In order to do that, they generalized Bourgain's ordinal index "measuring" how present is a given basic sequence in a Banach space, by using vines instead of trees.
After some reminders, we will show with similar arguments that alike families can capture the Lipschitz presence of ℓ_p, $1 \leq p < \infty$. This will allow us to deduce that a Banach space containing a bi-Lipschitz copy of every reflexive asymptotic-ℓ_1 space contains a Lipschitz copy of ℓ_1. Once this result is proved, we will see that the same can be done with coarse-Lipschitz embeddings and that a Banach space contains a coarse-Lipschitz copy of c_0 if it contains a coarse-Lipschitz copy of every reflexive asymptotic-c_0 space. Finally, we will prove the following equivalences.

Proposition. *Let X be a Banach space, $1 \leq p < \infty$. The following are equivalent:*

(i) $I^{Lip}_{\ell_p}(X) > \omega$;

(ii) $I^{cL}_{\ell_p}(X) > \omega$;

(iii) ℓ_p *is finitely representable in* X.

Similarly, $I^{Lip}_{c_0}(X) > \omega$ *if and only if* $I^{cL}_{c_0}(X) > \omega$ *if and only if* c_0 *is finitely representable in* X.

where $\mathrm{I}^{\mathrm{Lip}}_{\ell_p}(X)$ (respectively $\mathrm{I}^{\mathrm{cL}}_{\ell_p}(X)$) is the ordinal index measuring the Lipschitz presence (respectively coarse Lipschitz presence) of ℓ_p in X.

3. Notation

All Banach spaces are over the field $\mathbb{K}$, which is either $\mathbb{R}$ or $\mathbb{C}$. We denote the closed unit ball of a Banach space X by B_X, and its unit sphere by S_X. Given a Banach space X with norm $\|\cdot\|_X$, we simply write $\|\cdot\|$ as long as it is clear from the context on which space it is defined. By *subspace*, we shall always mean closed subspace. Unless otherwise specified, all spaces are assumed to be infinite-dimensional. Throughout, we let Ban denote the class of all Banach spaces over $\mathbb{K}$. We let Sep denote the class of separable members of Ban. We recall that a Banach space is said to be *quasi-reflexive* if the image of its canonical embedding into its bidual is of finite codimension in its bidual.
We say that a basic sequence $(e_i)_{i\in\mathbb{N}}$ of a Banach space E is *c-unconditional*, for some $c \geq 1$, if, for any $(a_i)_{i\in\mathbb{N}}, (b_i)_{i\in\mathbb{N}} \in c_{00}$ (the vector space of all real sequences with finite support), we have :

$$\Big\|\sum_{i=1}^{\infty} a_i e_i\Big\| \leq c\Big\|\sum_{i=1}^{\infty} b_i e_i\Big\|$$

whenever $|a_i| \leq |b_i|$ for every $i \in \mathbb{N}$.
Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces. Let $\mathcal{E} = (e_n)_{n\in\mathbb{N}}$ be a 1-unconditional basic sequence in a Banach space E with norm $\|\cdot\|_E$. We define the sum $\left(\sum_{n\in\mathbb{N}} X_n\right)_{\mathcal{E}}$ to be the space of sequences $(x_n)_{n\in\mathbb{N}}$, where $x_n \in X_n$ for all $n \in \mathbb{N}$, such that $\sum_{n\in\mathbb{N}} \|x_n\|_{X_n} e_n$ converges in E, and we set

$$\|(x_n)_{n\in\mathbb{N}}\| = \Big\|\sum_{n\in\mathbb{N}} \|x_n\|_{X_n} e_n\Big\|_E < \infty.$$

One can check that $\left(\sum_{n\in\mathbb{N}} X_n\right)_{\mathcal{E}}$, endowed with the norm $\|\cdot\|$ defined above, is a Banach space. We can, in a similar way, define finite sums $\left(\sum_{j=1}^{n} X_j\right)_{\mathcal{E}}$ for all $n \in \mathbb{N}$, and, in case $n = 2$, we will write $X_1 \bigoplus_{\mathcal{E}} X_2$. If it is implicit what is the basis $\mathcal{E}$ of the Banach space E that we are working with, we write $\left(\sum_{n\in\mathbb{N}} X_n\right)_E$ or $X_1 \bigoplus_{E} X_2$. Also, if the X_n's are all the same, say $X_n = X$, for all $n \in \mathbb{N}$, we write $E(X)$.

Let us finish this section with the following definition.
Let $p \in (1, \infty)$ and E be a Banach space with a 1-unconditional basis $(e_n)_{n\in\mathbb{N}}$. We say that the basis $(e_n)_{n\in\mathbb{N}}$ is *p-convex with convexity constant* C if :

$$\Big\|\sum_{j\in\mathbb{N}} (|x_j^1|^p + \cdots + |x_j^k|^p)^{\frac{1}{p}} e_j\Big\|^p \leq C^p \sum_{n=1}^{k} \|x^n\|^p$$

for all $x^1 = \sum_{j=1}^{\infty} x_j^1 e_j, \cdots, x^k = \sum_{j=1}^{\infty} x_j^k e_j \in E$ (cf Definition 1.d.3 of [76]).

Chapter 1

Non-linear maps and asymptotic properties of Banach spaces

The goal of this chapter is to introduce the notions we will need in this thesis. In particular, we describe in details four asymptotic smoothness properties of Banach spaces, denoted $\mathsf{T}_p, \mathsf{A}_p, \mathsf{N}_p$, and P_p. We complete their description by proving the missing renorming characterization for A_p.

Let us briefly describe the organization of this chapter. In a first short section, we will define the categories of non-linear maps considered later, starting with metric equivalences and finishing with metric embeddings.

In a second section, based on a joint work with R. Causey and G. Lancien (see [30]), we start by recalling the definition and some basic facts about the Szlenk index. Next, we define the four asymptotic properties at stake $\mathsf{T}_p, \mathsf{A}_p, \mathsf{N}_p$, and P_p, first in terms of certain games and then in terms of weakly null trees. Then, we give a complete description of these classes, insisting on the renorming characterizations. For the property A_p, this result is new and will be used in Chapter 3 to prove the stability of this class under coarse-Lipschitz equivalences. We finish this chapter with a proof of the separable determination of all these properties.

1.1 Non-linear maps

1.1.1 Metric equivalences

Definition 1.1.1. Let (M,d) and (N,δ) be two metric spaces. A map $f : M \to N$ is a *Lipschitz equivalence (or Lipschitz isomorphism)* from M to N if f is a Lipschitz bijection from M to N with Lipschitz inverse. If there exists a Lipschitz equivalence from M to N, we say that M and N are *Lipschitz equivalent (or Lipschitz isomorphic)* and we denote $M \overset{L}{\sim} N$.

Definition 1.1.2. Let (M,d) and (N,δ) be two unbounded metric spaces and $f : M \to N$ be a map. We say that f is *coarse Lipschitz* if there exist $A, B \in [0,\infty)$ such that

$$\forall x,y \in M, \quad \delta(f(x),f(y)) \leq Ad(x,y) + B.$$

We say that f is a *coarse Lipschitz equivalence* from M to N, if it is coarse Lipschitz and there exist a coarse Lipschitz map $g : N \to M$ and a constant $C \geq 0$ such that

$$\forall x \in M \quad d\big((g \circ f)(x), x\big) \leq C \quad \text{and} \quad \forall y \in N \quad \delta\big((f \circ g)(y), y\big) \leq C.$$

If there exists a coarse Lipschitz equivalence from M to N, we say that M and N are *coarse Lipschitz equivalent* and denote $M \overset{CL}{\sim} N$.

This notion of coarse Lipschitz equivalent metric spaces is the same as the notion of quasi-isometric metric spaces introduced by Gromov in [48] (see also the book [42] by E. Ghys and P. de la Harpe or the book [85] by P. W. Nowak and G. Yu).

We now turn to the notion of net in a metric space.

Definition 1.1.3. Let $0 < a \leq b$. An (a, b)-*net* in the metric space (M, d) is a subset $\mathcal{M}$ of M such that for every $z \neq z'$ in $\mathcal{M}$, $d(z, z') \geq a$ and for every x in M, $d(x, \mathcal{M}) < b$. Then a subset $\mathcal{M}$ of M is a *net* in M if it is an (a, b)-net for some $0 < a \leq b$.

Let us now give two technical equivalent formulations of the notion of coarse Lipschitz equivalence between Banach spaces. We refer to [32] or [47] for details.

Proposition 1.1.4. *Let X and Y be two Banach spaces and let $f : X \to Y$ be a map. The following assertions are equivalent.*

(i) The map f is a coarse Lipschitz equivalence.

(ii) There exist $A_0 > 0$ and $K \geq 1$ such that for all $A \geq A_0$ and all maximal A-separated subset $\mathcal{M}$ of X, $\mathcal{N} = f(\mathcal{M})$ is a net in Y and

$$\forall x, x' \in \mathcal{M} \quad \frac{1}{K}\|x - x'\| \leq \|f(x) - f(x')\| \leq K\|x - x'\|.$$

(iii) There exist two <u>continuous</u> coarse Lipschitz maps $\varphi : X \to Y$ and $\psi : Y \to X$ and a constant $C \geq 0$ such that $\|\varphi(x) - f(x)\| \leq C$ for all x in X and

$$\forall x \in X \quad \|(\psi \circ \varphi)(x) - x\| \leq C \quad \text{and} \quad \forall y \in Y \quad \|(\varphi \circ \psi)(y) - y\| \leq C.$$

1.1.2 Metric embeddings

Let us give some definitions on metric embeddings.
Let (X, d_X) and (Y, d_Y) be two metric spaces, let f be a map from X to Y.
We define the *compression modulus* of f by

$$\forall t \geq 0, \;\; \rho_f(t) = \inf\{d_Y(f(x), f(y)); d_X(x, y) \geq t\};$$

and the *expansion modulus* of f by

$$\forall t \geq 0, \;\; \omega_f(t) = \sup\{d_Y(f(x), f(y)); d_X(x, y) \leq t\}.$$

We adopt the convention $\inf(\varnothing) = +\infty$. Note that, for every $x, y \in X$, we have

$$\rho_f(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq \omega_f(d_X(x, y)).$$

We say that f is a *bi-Lipschitz embedding* if there exist A, B in $(0, \infty)$ such that $\rho_f(t) \geq At$ and $\omega_f(t) \leq Bt$ for all $t \geq 0$. If there exists such an embedding f, we denote $(X, d_X) \underset{L}{\hookrightarrow} (Y, d_Y)$.
If the metric spaces are unbounded, the map f is said to be a *coarse embedding* if $\lim_{t\to\infty} \rho_f(t) = \infty$ and $\omega_f(t) < \infty$ for all $t > 0$. It is worth mentioning that if the metric spaces are unbounded and X is metrically convex ($i.e$ for all $x, y \in X$, for every $\lambda \in (0, 1)$, there exists $z_\lambda \in X$ such that $d_X(x, z_\lambda) = \lambda d_X(x, y)$ and $d_X(y, z_\lambda) = (1 - \lambda)d_X(x, y)$), having $\omega_f(t) < \infty$ for all $t > 0$ implies the existence of two constants $A, B > 0$ so that $\omega_f(t) \leq At + B$ for every $t > 0$.
If one is given a family of metric spaces $(X_i)_{i\in I}$, one says that $(X_i)_{i\in I}$ *equi-Lipschitz embeds* into Y, denoted by $X_i \underset{eL}{\hookrightarrow} Y$, if there exist A, B in $(0, \infty)$ and, for all $i \in I$, maps $f_i : X_i \to Y$ such that $\rho_{f_i}(t) \geq At$ and $\omega_{f_i}(t) \leq Bt$ for all $t \geq 0$. One also says that the family $(X_i)_{i\in I}$ *equi-coarsely embeds* into Y if there exist non-decreasing functions $\rho, \omega : [0, \infty) \to [0, \infty)$ and for all $i \in I$, maps $f_i : X_i \to Y$ such that $\rho \leq \rho_{f_i}$, $\omega_{f_i} \leq \omega$, $\lim_{t\to\infty} \rho(t) = \infty$ and $\omega(t) < \infty$ for all $t > 0$.
Besides, we say that f is a *coarse Lipschitz embedding* if there exist A, B, C, D in $(0, \infty)$ such that $\rho_f(t) \geq At - C$ and $\omega_f(t) \leq Bt + D$ for all $t \geq 0$. If X and Y are Banach spaces, this is equivalent to the existence of numbers $\theta \geq 0$ and $0 < c_1 < c_2$ so that :

$$c_1\|x - y\|_X \leq \|f(x) - f(y)\|_Y \leq c_2\|x - y\|_X$$

for all $x, y \in X$ satisfying $\|x - y\|_X \geq \theta$.
If one is given a family of metric spaces $(X_i)_{i\in I}$, one says that $(X_i)_{i\in I}$ *equi-coarse-Lipschitz embeds* into Y if there exist A, B, C, D in $(0, \infty)$ and, for all $i \in I$, maps $f_i : X_i \to Y$ such that $\rho_{f_i}(t) \geq At - C$ and $\omega_{f_i}(t) \leq Bt + D$ for all $t \geq 0$.
Finally, a way to refine the scale of coarse embeddings is to talk about compression exponents, introduced by Guentner and Kaminker in [49]. Let X and Y to Banach spaces. The *compression exponent of* X *in* Y, denoted by $\alpha_Y(X)$, is the supremum of all $\alpha \in [0, 1)$ for which there exist a coarse embedding $f : X \to Y$ and A, C in $(0, \infty)$ so that $\rho_f(t) \geq At^\alpha - C$ for all $t > 0$.

Let us now introduce the notions related to asymptotic smoothness we will study in this thesis.

1.2 Szlenk index and asymptotic properties

This section is part of a joint work with Ryan Causey and Gilles Lancien (see [30]).

1.2.1 Definitions and first properties

Let us first recall the definition of the Szlenk derivation. For a Banach space X, $K \subset X^*$ weak*-compact, and $\varepsilon > 0$, we let $s_\varepsilon(K)$ denote the set of $x^* \in K$ such that for each weak*-neighborhood V of x^*, $\text{diam}(V \cap K) \geq \varepsilon$. For $1 \leqslant q < \infty$, we say X has *q-summable Szlenk index* provided that there exists a constant $c > 0$ such that for any $n \in \mathbb{N}$ and any $\varepsilon_1, \ldots, \varepsilon_n \geqslant 0$ such that $s_{\varepsilon_1} \ldots s_{\varepsilon_n}(B_{X^*}) \neq \varnothing$ (we write $s_{\varepsilon_1} \ldots s_{\varepsilon_n}(B_{X^*})$

instead of $s_{\varepsilon_1}(\ldots(s_{\varepsilon_n}(B_{X^*})))$ for convenience), $\sum_{i=1}^n \varepsilon_i^q \leqslant c^q$. In the $q=1$ case, we refer to this as *summable Szlenk index* rather than 1-summable Szlenk index.

We now recall the definition of the Szlenk index, based on the Szlenk derivation. For a Banach space X, $K \subset X^*$ weak*-compact, and $\varepsilon > 0$, we define the transfinite derivations

$$s_\varepsilon^0(K) = K,$$

$$s_\varepsilon^{\xi+1}(K) = s_\varepsilon(s_\varepsilon^\xi(K)),$$

and if ξ is a limit ordinal,

$$s_\varepsilon^\xi(K) = \bigcap_{\zeta<\xi} s_\varepsilon^\zeta(K).$$

For convenience, we let $s_0(K) = K$. If there exists an ordinal ξ such that $s_\varepsilon^\xi(K) = \varnothing$, we let $Sz(K,\varepsilon)$ denote the minimum such ordinal, and otherwise we write $Sz(K,\varepsilon) = \infty$. We let $Sz(K) = \sup_{\varepsilon>0} Sz(K,\varepsilon)$, where $Sz(K) = \infty$ if $Sz(K,\varepsilon) = \infty$ for some $\varepsilon > 0$. We let $Sz(X,\varepsilon) = Sz(B_{X^*},\varepsilon)$ and $Sz(X) = Sz(B_{X^*})$.

We define similarly the convex Szlenk index of X, denoted $Cz(X)$ and introduced in [46], from the following somewhat slower derivation: if $K \subset X^*$ is weak*-compact and $\varepsilon > 0$, then $c_\varepsilon(K)$ is the weak*-closed convex hull of $s_\varepsilon(K)$.

In Chapters 1 and 3, we will exclusively be concerned with Banach spaces X such that $Sz(X) \leqslant \omega$, where ω is the first infinite ordinal. Since $Sz(X) = 1$ if and only if X has finite dimension, and otherwise $Sz(X) \geqslant \omega$, we will actually only be concerned with the case $Sz(X) = \omega$. By compactness, $Sz(X) \leqslant \omega$ if and only if $Sz(X,\varepsilon)$ is a natural number for each $\varepsilon > 0$. We note that $Sz(X) < \infty$ if and only if X is *Asplund*. One characterization of Asplund spaces is that every separable subspace has a separable dual.

We recall that for any Banach space X and $0 < \varepsilon, \delta < 1$,

$$Sz(X,\varepsilon\delta) \leqslant \max\{Sz(X,\varepsilon)Sz(X,\delta); Sz(X,\delta)Sz(X,\varepsilon)\}.$$

From this it follows that if $Sz(X,\varepsilon)$ is a natural number for each $\varepsilon > 0$, the Szlenk power type

$$\mathsf{p}(X) := \lim_{\varepsilon\to 0^+} \frac{\log Sz(X,\varepsilon)}{|\log(\varepsilon)|}$$

is finite. It also holds that for any ordinal ξ, any $\varepsilon > 0$, and any natural number n, if $Sz(X,\varepsilon) > \xi$, then $Sz(X,\frac{\varepsilon}{n}) > \xi n$. Indeed, this follows from realizing

$$B_{X^*} = \frac{1}{n}B_{X^*} + \ldots + \frac{1}{n}B_{X^*}$$

and noting that the $\frac{\varepsilon}{n}$-derivations act on one summand at a time in the same way that the ε-derivations act on B_{X^*}. Therefore the ε-Szlenk index grows subgeometrically but superarithmetically. The superarithmetic growth implies that for any infinite dimensional Banach space, $\mathsf{p}(X) \geqslant 1$.

Let us introduce the *modulus of asymptotic uniform smoothness* of X. If X is infinite-dimensional, for $\sigma \geqslant 0$, we define

$$\overline{\rho}_X(\sigma) = \sup_{y\in B_X} \inf_{E\in \mathrm{cof}(X)} \sup_{x\in B_E} \|y+\sigma x\| - 1,$$

where cof(X) denotes the set of finite codimensional subspaces of X. For the sake of completeness, we define $\overline{\rho}_X(\sigma) = 0$ for all $\sigma \geqslant 0$, when X is finite-dimensional. We note that

$$\overline{\rho}_X(\sigma) = \sup_{y \in B_X} \sup\{\limsup_{\lambda} \|y + \sigma x_\lambda\| - 1 : (x_\lambda) \subset B_X \text{ is a weakly null net}\}.$$

It follows easily from the triangle inequality that $\overline{\rho}_X$ is a convex function. Since $\overline{\rho}_X(0) = 0$, we deduce that $\sigma \mapsto \frac{\overline{\rho}_X(\sigma)}{\sigma}$ is non-decreasing on $(0, \infty)$. Therefore

$$\inf_{\sigma > 0} \frac{\overline{\rho}_X(\sigma)}{\sigma} = \lim_{\sigma \to 0^+} \frac{\overline{\rho}_X(\sigma)}{\sigma}.$$

We say X is *asymptotically uniformly smooth* (in short *AUS*) if

$$\inf_{\sigma > 0} \frac{\overline{\rho}_X(\sigma)}{\sigma} = 0.$$

We say X is *asymptotically uniformly smoothable* (*AUS-able*) if X admits an equivalent AUS norm. For $1 < p < \infty$, we say X is *p-asymptotically uniformly smooth* (in short *p-AUS*) if

$$\sup_{\sigma > 0} \frac{\overline{\rho}_X(\sigma)}{\sigma^p} < \infty.$$

We say X is *p-asymptotically uniformly smoothable* (*p-AUS-able*) if X admits an equivalent p-AUS norm. We say X is *asymptotically uniformly flat* (*AUF*) if there exists $\sigma_0 > 0$ such that $\overline{\rho}_X(\sigma_0) = 0$. We say X is *asymptotically uniformly flattenable* if X admits an equivalent AUF norm. Of course, p-AUS spaces and AUF spaces are AUS spaces.

We also define the two following moduli. For $\varepsilon > 0$, we let

$$\overline{\theta}^*_X(\varepsilon) = \sup\{\delta \geq 0,\ s_\varepsilon(B_{X^*}) \subset (1-\delta)B_{X^*}\}$$

and we define the *modulus of weak* asymptotic uniform convexity* by

$$\overline{\delta}^*_X(t) = \inf_{x^* \in S_{X^*}} \sup_{E} \inf_{y^* \in S_E} \|x^* + ty^*\| - 1$$

where E runs through the set of all weak*-closed subspaces of X^* of finite codimension.

It is well known that the dual Young function of the modulus of asymptotic uniform smoothness is equivalent to the so-called modulus of weak* asymptotic uniform convexity $\overline{\delta}^*_X$ (see Proposition 2.1 and Corollary 2.3 in [34]), where two continuous increasing functions f, g on $[0, 1]$ satisfying $f(0) = g(0) = 0$ are said to be equivalent if one can find $C > 0$ such that $f(t) \geq g(t/C)$ and $g(t) \geq f(t/C)$ for all $t \in [0, 1]$. Let us note the following.

Proposition 1.2.1. *The modulus $\overline{\delta}^*_X$ is equivalent to $\overline{\theta}^*_X$.*

Proof. First, let us remark that the inequality $\overline{\theta}^*_X(t) \geq \overline{\delta}^*_X(t/2)$ is clear. Let us prove that $\overline{\theta}^*_X(t/2) \leq \overline{\delta}^*_X(t)$. Let $1 \geq \delta > \overline{\delta}^*_X(t)$.
There exists $z^* \in S_{X^*}$ such that for every weak*-closed subspace E of X^* of finite codimension, we can find $y^* \in S_E$ so that $\|z^* + ty^*\| \leq 1 + \delta$. Then it is enough to note that $x^* = \frac{1}{1+\delta} z^* \in B_{X^*}$ satisfies $\|x^*\| > 1 - \delta$ and $x^* \in s_{t/2}(B_{X^*})$ since $\delta \leq 1$. □

It will be more convenient for us to work with $\overline{\theta}^*_X$. We shall only need the following version of Proposition 2.1 in [34].

Proposition 1.2.2. *There exists a universal constant $C \geq 1$ such that for any Banach space X and any $0 < \sigma, \tau < 1$,*

1. *If $\overline{\rho}_X(\sigma) < \sigma\tau$, then $\overline{\theta}^*_X(C\tau) \geq \sigma\tau$.*
2. *If $\overline{\theta}^*_X(\tau) > \sigma\tau$, then $\overline{\rho}_X(\frac{\sigma}{C}) \leq \sigma\tau$.*

For $1 \leqslant q < \infty$, a Banach space X, and a sequence $(x_i)_{i=1}^\infty \subset X$, we define the (possibly infinite) quantity

$$\|(x_i)_{i=1}^\infty\|_q^w = \sup\{\|(x^*(x_i))_{i=1}^\infty\|_{\ell_q} : x^* \in B_{X^*}\}.$$

We also define this quantity for finite sequences,

$$\|(x_i)_{i=1}^n\|_q^w = \sup\{\|(x^*(x_i))_{i=1}^n\|_{\ell_q^n} : x^* \in B_{X^*}\}.$$

Note that, if $p \in (1, \infty]$ is the conjugate exponent of q, we have that

$$\|(x_i)_{i=1}^\infty\|_q^w = \inf\big\{c \in (0, \infty],\ \forall a = (a_i)_{i=1}^\infty \in \ell_p\ \|\sum_{i=1}^\infty a_i x_i\| \leq c\|a\|_p\big\}.$$

A similar formula is valid for $\|(x_i)_{i=1}^n\|_q^w$.

We next define four different two-players games on a Banach space X. Fix $1 < p \leqslant \infty$ and let $1/p + 1/q = 1$. For $c > 0$ and $n \in \mathbb{N}$, we define the $T(c,p)$ game on X, the $A(c,p,n)$ game, and the $N(c,p,n)$ game. Let D be a weak neighborhood base at 0 in X. In the $T(c,p)$ game, Player I chooses a weak neighborhood $U_1 \in D$, and Player II chooses $x_1 \in U_1 \cap B_X$. Player I chooses $U_2 \in D$, and Player II chooses $x_2 \in U_2 \cap B_X$. Play continues in this way until $(x_i)_{i=1}^\infty$ has been chosen. Player I wins if $\|(x_i)_{i=1}^\infty\|_q^w \leqslant c$, and Player II wins otherwise.

The $A(c,p,n)$ game is similar, except the game terminates after the n^{th} turn. Player I wins if $\|(x_i)_{i=1}^n\|_q^w \leqslant c$, and Player II wins otherwise.

In the $N(c,p,n)$ game, as in the $A(c,p,n)$ game, the game terminates after the n^{th} turn. Player I wins if $\left\|\sum_{i=1}^n x_i\right\| \leqslant cn^{1/p}$, and Player II wins otherwise.

Finally, in the $\Theta(c,n)$ game, Player I wins if $\left\|\sum_{i=1}^n x_i\right\| \leqslant c$, and Player II wins otherwise.

It is known (see [26], Section 3) that each of these games is determined. That is, in each game, either Player I or Player II has a winning strategy. We let $\mathsf{t}_p(X)$ denote the infimum of $c > 0$ such that Player I has a winning strategy in the $T(c,p)$ game, provided such a c exists, and we let $\mathsf{t}_p(X) = \infty$ otherwise. We let $\mathsf{a}_{p,n}(X)$ denote the infimum of $c > 0$ such that Player I has a winning strategy in the $A(c,p,n)$ game, and we let $\mathsf{a}_p(X) = \sup_n \mathsf{a}_{p,n}(X)$. We note that $\mathsf{a}_p(X)$ is the infimum of $c > 0$ such that for each $n \in \mathbb{N}$, Player I has a winning strategy in the $A(c,p,n)$ game if such a c exists, and $\mathsf{a}_p(X) = \infty$ otherwise. We let $\theta_n(X)$ denote the infimum of $c > 0$ such that Player I has a winning strategy in the $\Theta(c,n)$ game, noting that $\theta_n(X) \leqslant n$. Finally, we let $\mathsf{n}_{p,n}(X) = \theta_n(X)/n^{1/p}$ and $\mathsf{n}_p(X) = \sup_n \mathsf{n}_{p,n}(X)$, noting that $\mathsf{n}_p(X)$ is the infimum of $c > 0$ such that for each $n \in \mathbb{N}$, Player I has a winning strategy in the $N(c,p,n)$ game, provided such a c exists, and $\mathsf{n}_p(X) = \infty$ otherwise.

Remark 1.2.3. The existence of winning strategies, and therefore the constants associated with these games, do not depend upon the particular choice of the weak neighborhhod base D. Therefore in the case that X has a separable dual, these constants are sequentially determined. Let us indicate the argument.

Proof. If D_1, D_2 are two weak neighborhood bases at 0 in X, and Player I has a winning strategy in any of the games above when Player I is required to choose from D_1, then this winning strategy can be used to construct a winning strategy choosing from D_2 by choosing at each stage of the game any member of D_2 which is a subset of the member of D_1 indicated by the winning strategy. From this it follows that the values of the associated constants also do not depend on D. In particular, in the case that X^* is separable, we can use a fixed countable, linearly ordered weak neighborhood base D. □

Let $D^{\leqslant n} = \cup_{i=1}^n D^i$. Let $D^{<\omega} = \cup_{i=1}^\infty D^i$, and let D^ω denote the set of all infinite sequences whose members lie in D. Let $D^{\leqslant\omega} = D^{<\omega} \cup D^\omega$. For $s, t \in D^{<\omega}$, we let $s \frown t$ denote the concatenation of s with t. We let $|t|$ denote the length of t. For $0 \leqslant i \leqslant |t|$, we let $t|_i$ denote the initial segment of t having length i, where $t|_0 = \varnothing$ is the empty sequence. If $s \in \{\varnothing\} \cup D^{<\omega}$, we let $s \prec t$ denote the relation that s is a proper initial segment of t.

We say a function $\varphi : D^{<\omega} \to D^{<\omega}$ is a *pruning* provided that

(i) $|\varphi(t)| = |t|$ for all $t \in D^{\leqslant n}$,

(ii) if $s \prec t$, then $\varphi(s) \prec \varphi(t)$,

(iii) if $\varphi((U_1, \dots, U_k)) = (V_1, \dots, V_k)$, then $V_k \subset U_k$.

We define prunings $\varphi : D^{\leqslant n} \to D^{\leqslant n}$ similarly.

Given D a weak neighborhood base of 0 in X and $(x_t)_{t\in D^{<\omega}} \subset X$, we say $(x_t)_{t\in D^{<\omega}}$ is

(i) *weakly null of type* I provided that for each $t = (U_1, \dots, U_k)$, $x_t \in U_k$,

(ii) *weakly null of type* II provided that for each $t \in \{\varnothing\} \cup D^{<\omega}$, $(x_{t\frown(U)})_{U\in D}$ is a weakly null net. Here D is directed by reverse inclusion.

The notions of weakly null of types I and II for collections indexed by $D^{\leqslant n}$ are defined similarly. Note that a weakly null collection of type I is weakly null of type II. We now link these notions with our various games.

Proposition 1.2.4. *Let X be a Banach space, let $p \in (1, \infty]$, and $c > 0$. Then, Player II has a winning strategy in the $T(c, p)$ game on X if and only if there exists a collection $(x_t)_{t\in D^{<\omega}} \subset B_X$ such that*

(a) $(x_t)_{t\in D^{<\omega}}$ is weakly null of type I, and

(b) for each $\tau \in D^\omega$, $\|(x_{\tau|_i})_{i=1}^\infty\|_q^w > c$.

Proof. First, if such a collection exists, we can use it to define a winning strategy for Player II in the $T(c, p)$ game. When Player I chooses $U_1 \in D$, then Player II chooses $x_{(U_1)}$. Player I chooses $U_2 \in D$, to which Player II's response is $x_{(U_1,U_2)}$. Play continues in this way, and the result is $(x_{\tau|_i})_{i=1}^\infty$ for some $\tau \in D^\omega$, which satisfies $\|(x_{\tau|_i})_{i=1}^\infty\|_q^w > c$. On the other hand, if Player II has a winning strategy in the $T(c, p)$ game, we define by induction on k the vector $x_{(U_1,\dots,U_k)}$ to be Player II's response according to this winning strategy following the choices $U_1, x_{(U_1)}, U_2, x_{(U_1,U_2)}, \dots, U_k$ in the $T(c,p)$ game. It follows from the rules of the game that (a) is satisfied, and it follows from the fact that Player II plays according to a winning strategy that (b) is satisfied. $\square$

Analogous statements about $(x_t)_{t\in D^{\leqslant n}} \subset B_X$ can be made for the $A(c,p,n)$, $N(c,p,n)$, and $\Theta(c,n)$ games. We also have

Proposition 1.2.5. *Let X be a Banach space, let $p \in (1,\infty]$, and $c > 0$. Then, Player II has a winning strategy in the $T(c,p)$ game if and only if there exists a collection $(x_t)_{t\in D^{<\omega}} \subset B_X$ such that*

(a) $(x_t)_{t\in D^{<\omega}}$ is weakly null of type II, and

(b) for each $\tau \in D^\omega$, $\|(x_{\tau|_i})_{i=1}^\infty\|_q^w > c$.

Proof. Since any collection which is weakly null of type I is also weakly null of type II, by the previous proposition, it is sufficient to note that if $(x_t)_{t\in D^{<\omega}} \subset B_X$ is weakly null of type II, then there exists a pruning $\varphi : D^{<\omega} \to D^{<\omega}$ such that $(x_{\varphi(t)})_{t\in D^{<\omega}} \subset B_X$ is weakly null of type I. Moreover, property (b) is retained by the collection $(x_{\varphi(t)})_{t\in D^{<\omega}}$. $\square$

Again, analogous statements hold for collections indexed by $D^{\leqslant n}$ and the games $A(c,p,n)$, $N(c,p,n)$, and $\Theta(c,n)$. Unless otherwise specified, by a weakly null collection $(x_t)_{t\in D^{<\omega}}$ in X, we shall mean weakly null of type II. However, it might be convenient to use that we may assume it to be weakly null of type I.

Remark 1.2.6. As we already mentioned, in the case that X^* is separable, we can use a fixed countable, linearly ordered weak neighborhood base D and, by identifying D with $\mathbb{N}$, characterize the constants $\mathsf{t}_p(X), \mathsf{a}_{p,n}(X), \mathsf{a}_p(X), \theta_n(X), \mathsf{n}_{p,n}(X), \mathsf{n}_p(X)$ using trees indexed by $\mathbb{N}^{<\omega}$ or $\mathbb{N}^{\leqslant n}$ rather than $D^{<\omega}$ or $D^{\leqslant n}$. Now, if X is separable and there exists some $1 < p \leqslant \infty$ such that any of the constants $\mathsf{t}_p(X), \mathsf{a}_p(X), \mathsf{n}_p(X)$ is finite, then $Sz(X) \leqslant \omega$, X is Asplund, and X^* is separable. However, we can use $\mathbb{N}^{<\omega}$ (resp. $\mathbb{N}^{\leqslant n}$) in place of $D^{<\omega}$ (resp. $D^{\leqslant n}$) as index sets to compute the values of these constants only if we already know that X^* is separable, because of examples like ℓ_1 with the Schur property. So, for example, once we know $\mathsf{p}(X)$ is finite, we can characterize its value using trees indexed by $\mathbb{N}^{<\omega}$, but we cannot use trees indexed by $\mathbb{N}^{<\omega}$ to determine whether $\mathsf{t}_p(X)$ is finite.

We conclude this subsection with elementary statements that we shall use to stabilize weakly null trees. Note that the operation described below is actually a pruning.

Proposition 1.2.7. *Let $(D, \leqslant_D)$ be any directed set, F a finite set, n a natural number, and $f : D^n \to F$ a function. There exists $\theta : D^{\leqslant n} \to D^{\leqslant n}$ preserving lengths and initial segments such that*

(i) if $\theta((U_1,\dots,U_k)) = (V_1,\dots,V_k)$, then $U_k \leqslant_D V_k$,

(ii) $f \circ \theta|_{D^n}$ is constant.

Proof. We work by induction. For $x \in F$, let $I_x = \{U \in D : f((U)) = x\}$. Since $\cup_{x\in F} I_x = D$ and F is finite, there exists some $x \in F$ such that I_x is cofinal in D. This means that for any $U \in D$, there exists $V_U \in I_x$ such that $U \leqslant_D V_U$. Define $\theta((U)) = (V_U)$ and note that $f \circ \theta|_{D^1} \equiv x$.

Next, assume the result holds for some n and fix $f : D^{n+1} \to F$. For each $U \in D$, define $f_U : D^n \to F$ by $f_U((U_1, \ldots, U_n)) = f((U, U_1, \ldots, U_n))$. By the inductive hypothesis, there exist $\theta_U : D^{\leqslant n} \to D^{\leqslant n}$ which preserves lengths and initial segments and satisfying (i) and (ii). Fix $x_U \in F$ such that $f_U \circ \theta_U|_{D^n} \equiv x_U$. Define $g : D^1 \to F$ by $g((U)) = x_U$. By the base case, there exists $\phi : D^1 \to D^1$ satisfying (i) and (ii) with f replaced by g. Define $\theta : D^{\leqslant n+1} \to D^{\leqslant n+1}$ by $\theta((U)) = \phi(U)$ and $\theta((U, U_1, \ldots, U_k)) = \phi(U) \frown \theta_{\phi(U)}(U_1, \ldots, U_k)$. □

Corollary 1.2.8. *Let $(D, \leqslant_D)$ be any directed set, (K, d) a totally bounded metric space, n a natural number, and $f : D^n \to K$ a function. For any $\varepsilon > 0$, there exist $\theta : D^{\leqslant n} \to D^{\leqslant n}$ preserving lengths and initial segments and a subset B of K of diameter less than ε such that*

(i) if $\theta((U_1, \ldots, U_k)) = (V_1, \ldots, V_k)$, then $U_k \leqslant_D V_k$,

(ii) $f(\theta(t)) \in B$ for all $t \in D^n$.

Proof. Let $B_1, \ldots, B_m$ be a cover of K by sets of diameter less than ε. Define $g : D^n \to \{1, \ldots, m\}$ by letting $g(t) = \min\{i \leqslant m : f(t) \in B_i\}$. Apply Proposition 1.2.7 to g. □

1.2.2 The properties and their relations

For $1 < p \leqslant \infty$, we let T_p denote the class of Banach spaces X such that $\mathsf{t}_p(X) < \infty$. The classes A_p and N_p are defined similarly using a_p and n_p. We let $\mathsf{P}_p = \bigcap_{1<r<p} \mathsf{T}_r$. We let D_1 denote the class of all Banach spaces the Szlenk index of which does not exceed ω. We now present the following alternative descriptions of each class. We have chosen to quickly indicate the easy arguments, to give precise references for others and to detail the new ones. We give this overview, insisting on the renorming characterizations, as they are crucial for our non-linear applications.

We start with the description of T_p. The next theorem is the main result from [26]. We briefly explain the easy implications and emphasize the key part of the proof.

Theorem 1.2.9. *Fix $1 < p \leqslant \infty$ and let q be conjugate to p. Let X be a Banach space. The following are equivalent*

(i) $X \in \mathsf{T}_p$.

(ii) There exists a constant $c > 0$ such that for any weak neighborhood base D at 0 in X and any weakly null $(x_t)_{t\in D^{<\omega}} \subset B_X$, there exists $\tau \in D^\omega$ such that $\|(x_{\tau|_i})_{i=1}^\infty\|_q^w \leqslant c$.

(iii) X is p-AUS-able (resp. AUF-able if $p = \infty$).

(iv) There exist an equivalent norm $|\cdot|$ on X and $c > 0$ such that for each $\varepsilon \in [0, 1]$, $s_\varepsilon(B_{X^*}^{|\cdot|}) \subset (1 - c\varepsilon^q)B_{X^*}^{|\cdot|}$. In other words, $\overline{\theta}_{|\cdot|}^*(\varepsilon) \geq c\varepsilon^q$

Proof. The equivalence between (i) and (ii) follows immediately from our discussions on winning strategies in the $T(c, p)$ game. More precisely, $\mathsf{t}_p(X)$ is the infimum of those c for which (ii) holds.

The equivalence between *(iii)* and *(iv)* is a immediate consequence of the duality Proposition 1.2.2.

We now detail the rather simple implication *(iii)* $\Rightarrow$ *(i)* and assume, as we may, that X is p-AUS. We look at $1 < p < \infty$ and $p = \infty$ separately. First consider the case $1 < p < \infty$. We note that $\sup_{\sigma>0} \overline{\rho}_X(\sigma)/\sigma^p < \infty$ if and only if there exists a constant $C \geqslant 1$ such that for any $x \in X$ and $\sigma \geqslant 0$, there exists a weak neighborhood U of 0 in X such that for any $y \in U \cap B_X$, $\|x + \sigma y\|^p \leqslant \|x\|^p + C^p\sigma^p + \varepsilon$. A finite net argument yields that for any compact $G \subset X$ and $\varepsilon > 0$, there exists a weak neighborhood U of 0 such that for any $x \in G$, any scalar b with $|b| \leqslant 1$, and any $y \in U \cap B_X$, $\|x + by\|^p \leqslant \|x\|^p + C^p|b|^p$. Using this fact, for $\varepsilon > 0$, we can define a winning strategy for Player I in the $T(C + \varepsilon, p)$ game by fixing $(\varepsilon_i)_{i=1}^\infty \subset (0, 1)$. Player I's initial choice U_1 is arbitrary. Once $U_1, x_1, \ldots, U_n, x_n$ have been chosen, let

$$G = \Big\{\sum_{i=1}^n b_i x_i : (b_i)_{i=1}^n \in B_{\ell_p^n}\Big\}$$

and choose U_{n+1} such that for any $x \in G$, $y \in U_{n+1} \cap B_X$, and any b with $|b| \leqslant 1$, $\|x + by\|^p \leqslant \|x\|^p + C^p|b|^p + \varepsilon_{n+1}$. This completes the recursive construction. For any $m \in \mathbb{N}$ and $(b_i)_{i=1}^m \in B_{\ell_p^m}$,

$$\begin{aligned}
\Big\|\sum_{i=1}^m b_i x_i\Big\|^p &\leqslant \Big\|\sum_{i=1}^{m-1} b_i x_i\Big\|^p + C^p|b_m|^p + \varepsilon_m \\
&\leqslant \Big\|\sum_{i=1}^{m-2} b_i x_i\Big\|^p + C^p|b_{m-1}|^p + C^p|b_m|^p + \varepsilon_{m-1} + \varepsilon_m \\
&\leqslant C^p \sum_{i=1}^m |b_i|^p + \sum_{i=1}^m \varepsilon_i \leqslant C^p + \sum_{i=1}^\infty \varepsilon_i.
\end{aligned}$$

If $\sum_{i=1}^\infty \varepsilon_i$ was chosen small enough, depending on the modulus of continuity of the function $t \mapsto t^{1/p}$ on $[0, C+1]$, this strategy is a winning strategy for Player I in the $T(C + \varepsilon, p)$ game. Therefore X has T_p. For the $p = \infty$ case, the argument is similar, except there exists a constant C such that for any $x \in X$ and $\sigma > 0$, there exists a weak neighborhood U of 0 such that for any $y \in U \cap B_X$, $\|x + \sigma y\| \leqslant \max\{\|x\|, C\sigma\}$.

Finally, we refer the reader to [26] for the difficult implication *(i)* $\Rightarrow$ *(iv)*.

$\square$

We now turn to the characterizations of A_p. Note that item *(iii)* is a completely new characterization. For that reason we recall the old arguments and detail the new ones. As we will see later, $\mathsf{A}_\infty = \mathsf{N}_\infty$, so we limit ourselves to $p \in (1, \infty)$ in the next statement.

Theorem 1.2.10. *Fix $1 < p < \infty$ and let q be conjugate to p. Let X be a Banach space. The following are equivalent*

(i) $X \in \mathsf{A}_p$.

(ii) There exists a constant $c > 0$ such that for any weak neighborhood base D at 0 in X, any $n \in \mathbb{N}$, and any weakly null $(x_t)_{t \in D^{\leqslant n}} \subset B_X$, there exists $t \in D^n$ such that $\|(x_{t|_i})_{i=1}^n\|_q^w \leq c$.

(iii) *There exist a constant $M \geq 1$ and a constant $C > 0$, such that for any $\tau \in (0,1]$ there exists a norm $|\ |$ on X satisfying $M^{-1}\|x\|_X \leq |x| \leq M\|x\|_X$ for all $x \in X$ and*

$$\forall \sigma \geq \tau, \quad \overline{\rho}_{|\ |}(\sigma) \leq C\sigma^p.$$

(iv) *X has q-summable Szlenk index.*

Proof. The equivalence between (i) and (ii) follows again from our initial discussion on games.

The implication $(ii) \Rightarrow (iii)$ is new. Let us prove it. Fix $1 < p < \infty$. Suppose that X is a Banach space and $a \geqslant 1$ is such that for each $n \in \mathbb{N}$ and $(x_t)_{t \in D^{\leqslant n}} \subset B_X$ weakly null, there exists $t \in D^n$ such that for all scalar sequences $(a_i)_{i=1}^n$,

$$\Big\| \sum_{i=1}^n a_i x_{t|_i} \Big\|^p \leqslant a^p \sum_{i=1}^n |a_i|^p.$$

We first note that for any $x \in X$, $n \in \mathbb{N}$, and $(x_t)_{t \in D^{\leqslant n}} \subset B_X$ weakly null, there exists $t \in D^n$ such that for all scalar sequences $(a_i)_{i=1}^n$,

$$\Big\| x + \sum_{i=1}^n a_i x_{t|_i} \Big\|^p \leqslant (2a)^p \Big[\|x\|^p + \sum_{i=1}^n |a_i|^p \Big]. \tag{1.1}$$

Indeed, for an appropriate branch t, it holds that

$$\begin{aligned} \Big\| x + \sum_{i=1}^n a_i x_{t|_i} \Big\|^p &\leqslant 2^p \max\Big\{ \|x\|^p, \Big\| \sum_{i=1}^n a_i x_{t|_i} \Big\|^p \Big\} \\ &\leqslant (2a)^p \max\Big\{ \|x\|^p, \sum_{i=1}^n |a_i|^p \Big\} \leqslant (2a)^p \Big[\|x\|^p + \sum_{i=1}^n |a_i|^p \Big]. \end{aligned}$$

Let now $A = 2a$. Set $f_0(x) = \frac{\|x\|}{A}$ and for $n \in \mathbb{N}$, define

$$f_n(x) = \Bigg[\sup_{(x_t)} \inf_{t} \sup_{(a_i)} \frac{1}{A^p} \Big\| x + \sum_{i=1}^n a_i x_{t|_i} \Big\|^p - \sum_{i=1}^n |a_i|^p \Bigg]^{1/p},$$

where the outer supremum is taken over all weakly null collections $(x_t)_{t \in D^{\leqslant n}}$ in B_X, the infimum is taken over $t \in D^n$, and the inner supremum is taken over all scalar sequences $(a_i)_{i=1}^n$. It follows from taking $x_t = 0$ for all t that $f_n(x) \geqslant \frac{\|x\|}{A}$ for all $x \in X$ and $n \in \mathbb{N}$. On the other hand, it follows from (1.1) that $f_n(x) \leqslant \|x\|$ for all $n \in \mathbb{N}$. We also have that $f_n(cx) = |c| f_n(x)$, for each $n \in \mathbb{N} \cup \{0\}$, each $x \in X$, and each scalar c. Let us detail this last fact. It is clear that $f_n(0) = 0$, so assume $c \neq 0$. It is also clear that $f_0(cx) = |c| f_0(x)$. Then we fix $n \in \mathbb{N}$, $x \in X$ and $n \in \mathbb{N}$. For an arbitrary $(x_t)_{t \in D^{\leqslant n}} \subset B_X$ weakly null and $b > f_n(x)$, there exists $t \in D^n$ such that for all $(a_i)_{i=1}^n$,

$$\frac{1}{A^p} \Big\| x + \sum_{i=1}^n a_i x_{t|_i} \Big\|^p - \sum_{i=1}^n |a_i|^p \leqslant b^p.$$

Then

$$\begin{aligned}&\frac{1}{A^p}\Big\|cx+\sum_{i=1}^n a_i x_{t|_i}\Big\|^p-\sum_{i=1}^n |a_i|^p\\&=|c|^p\Big[\frac{1}{A^p}\Big\|x+\sum_{i=1}^n c^{-1}a_i x_{t|_i}\Big\|^p-\sum_{i=1}^n |c^{-1}a_i|^p\Big]\leqslant |c|^p b^p.\end{aligned}$$

Since this holds for any $(x_t)_{t\in D^{\leqslant n}}\subset B_X$ weakly null, it holds that $f_n(cx)\leqslant |c|f_n(x)$. Repeating the argument, we deduce that $f_n(x)=f_n(c^{-1}cx)\leqslant |c|^{-1}f_n(cx)$, which gives the reverse inequality.

The key step will be to show the following: for each $n\in\mathbb{N}\cup\{0\}$, each weakly null net $(x_U)_{U\in D}\subset B_X$, and each $\sigma>0$,

$$\limsup_U f_n(x+\sigma x_U)^p\leqslant f_{n+1}(x)^p+\sigma^p. \tag{1.2}$$

So assume $\eta<\limsup_U f_n(x+\sigma x_U)^p$. By passing to a subnet and relabeling, we can assume $\eta<f_n(x+\sigma x_U)^p$ for all U. For each U, we find $(x_t^U)_{t\in D^{\leqslant n}}\subset B_X$ weakly null such that for each $t\in D^n$, there exists $(a_i)_{i=1}^n$ such that

$$\frac{1}{A^p}\Big\|x+\sigma x_U+\sum_{i=1}^n a_i x^U_{t|_i}\Big\|^p-\sum_{i=1}^n |a_i|^p>\eta.$$

We define the weakly null collection $(x_t)_{t\in D^{\leqslant n+1}}\subset B_X$ by letting $x_{(U)}=x_U$ and $x_{(U,U_1,\dots,U_k)}=x^U_{(U_1,\dots,U_k)}$ for $1\leqslant k\leqslant n$. By the definition of $f_{n+1}(x)^p$, for any $\varepsilon>0$, there exists $s\in D^{n+1}$ such that for all $(b_i)_{i=1}^{n+1}$,

$$\frac{1}{A^p}\Big\|x+\sum_{i=1}^{n+1} b_i x_{s|_i}\Big\|^p-\sum_{i=1}^{n+1}|b_i|^p\leqslant f_{n+1}(x)^p+\varepsilon.$$

Write $s=(U,U_1,\dots,U_n)$ and let $t=(U_1,\dots,U_n)$. Then there exists $(a_i)_{i=1}^n$ such that, combining this paragraph with the previous and letting $b_1=\sigma$ and $b_{i+1}=a_i$ for $1\leqslant i\leqslant n$, it holds that

$$\begin{aligned}\eta&<\frac{1}{A^p}\Big\|x+\sigma x_U+\sum_{i=1}^n a_i x^U_{s|_i}\Big\|^p-\sum_{i=1}^n|a_i|^p\\&=\frac{1}{A^p}\Big\|x+\sum_{i=1}^{n+1} b_i x_{s|_i}\Big\|^p-\sum_{i=1}^{n+1}|b_i|^p+\sigma^p\leqslant f_{n+1}(x)^p+\varepsilon+\sigma^p.\end{aligned}$$

Therefore $\eta<f_{n+1}(x)^p+\varepsilon+\sigma^p$. Since $\eta<\limsup_U f_n(x+\sigma x_U)^p$ and $\varepsilon>0$ were arbitrary, we have proved (1.2).

The next step is to average the f_n^p's. So, fix $N\in\mathbb{N}$ and define

$$g_N(x)^p=\frac{1}{N}\sum_{n=0}^{N-1} f_n(x)^p.$$

Clearly, we still have that for all $x \in X$ and $N \in \mathbb{N}$, $\frac{\|x\|}{A} \leqslant g_N(x) \leqslant \|x\|$ and $g_N(cx) = |c|g_N(x)$ for all scalars c. Then, applying (1.2) for each $n \in \{0, \dots, N-1\}$, we obtain that for any weakly null net $(x_U)_{U \in D} \subset B_X$, any $N \in \mathbb{N}$, and $x \in AB_X$,

$$\limsup_U g_N(x + \sigma x_U)^p \leqslant g_N(x)^p + \sigma^p + \frac{A^p}{N}. \tag{1.3}$$

The last stage of the proof is to "convexify" our function g_N. For that purpose, we set

$$|x|_N = \inf\Big\{\sum_{i=1}^n g_N(x_i) : n \in \mathbb{N}, x = \sum_{i=1}^n x_i\Big\},$$

which defines an equivalent norm on X satisfying $\frac{\|x\|}{A} \leqslant |x|_N \leqslant \|x\|$. Moreover, $B_X^{|\cdot|_N}$ is the closed, convex hull of $\{x \in X : g_N(x) < 1\}$. To see the non-trivial implication, it is enough to notice that if $x = \sum_{i=1}^n x_i \in X$ with $\sum_{i=1}^n g_N(x_i) < 1$, then

$$x = \sum_{i=1}^n \lambda_i y_i$$

where $y_i = \frac{\sum_{j=1}^n g_N(x_j)}{g_N(x_i)} x_i$ is such that $g_N(y_i) < 1$ and $\lambda_i = \frac{g_N(x_i)}{\sum_{j=1}^n g_N(x_j)} \in [0,1]$ for every $1 \le i \le n$, with $\sum_{i=1}^n \lambda_i = 1$.

We shall now prove that

$$\forall \sigma > 0, \quad \overline{\rho}_{(X,|\cdot|_N)}(\sigma) \leqslant \frac{A^p}{p}\big(\sigma^p + \frac{1}{N}\big). \tag{1.4}$$

First we fix $y \in X$ such that $g_N(y) < 1$. From this it follows that $\|y\| \leqslant A$. Fix $\sigma > 0$ and $(y_U)_{U \in D} \subset B_X^{|\cdot|_N}$ weakly null, define $x_U = A^{-1} y_U \in B_X$, so $(x_U)_{U\in D} \subset B_X$ is weakly null. Then we apply (1.3) to get

$$\begin{aligned}\limsup_U |y + \sigma y_U|_N^p &\leqslant \limsup_U g_N(y + \sigma A x_U)^p \\ &\leqslant g_N(y) + \sigma^p A^p + \frac{A^p}{N} < 1 + \sigma^p A^p + \frac{A^p}{N}.\end{aligned}$$

Therefore, thanks to the inequality $(1+t)^{1/p} \le 1 + t/p$,

$$\limsup_U |y + \sigma y_U|_N - 1 \leqslant \big(1 + \sigma^p A^p + \frac{A^p}{N}\big)^{1/p} - 1 \leqslant \frac{A^p}{p}\big(\sigma^p + \frac{1}{N}\big).$$

Next fix $x \in B_X^{|\cdot|_N}$. As noted above, $B_X^{|\cdot|_N}$ is the closed, convex hull of $\{y \in X : g_N(y) < 1\}$. Therefore for each $\varepsilon > 0$, we can find $y_1, \dots, y_k \in X$ with $g_N(y_i) < 1$ and convex coefficients $w_1, \dots, w_k$ such that $|x - \sum_{i=1}^k w_i y_i|_N < \varepsilon$. Then

$$\begin{aligned}\limsup_U |x + \sigma y_U|_N - 1 &\leqslant \varepsilon + \limsup_U \sum_{i=1}^k w_i(|y_i - \sigma y_U|_N - 1) \\ &\leqslant \varepsilon + \sum_{i=1}^k w_i\Big(\frac{A^p}{p}\big(\sigma^p + \frac{1}{N}\big)\Big) = \varepsilon + \frac{A^p}{p}\big(\sigma^p + \frac{1}{N}\big).\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this finishes the proof of (1.4).

Finally, it is clear, by taking N large enough in (1.4), that for any $\tau > 0$ there exists an equivalent norm $|\cdot|$ on X such that $\frac{\|x\|}{A} \leqslant |x| \leqslant \|x\|$ and for any $\sigma \geqslant \tau$, $\overline{\rho}_{(X,|\cdot|)}(\sigma) \leqslant \frac{A_1^p}{p}\sigma^p$. We have proved that X satisfies (iii).

Next we prove $(iii) \Rightarrow (iv)$, which is also new. So assume (iii) is satisfied. Then, it follows from Proposition 1.2.2 that there exists $\gamma \in (0,1]$ so that for any $t_0 \in (0,1]$ there exists a norm $|\ |$ on X satisfying

$$\forall x \in X,\ M^{-1}\|x\|_X \leq |x| \leq M\|x\|_X \ \text{ and } \ \forall t \in [t_0, 1],\ \overline{\theta}^*_{|\ |}(t) \geq \gamma t^q.$$

Fix now $\varepsilon_1, \dots, \varepsilon_n \in (0,1]$ and pick an equivalent norm $|\ |$ as above for $t_0 = \min\{\frac{\varepsilon_1}{4M^2}, \dots, \frac{\varepsilon_n}{4M^2}\}$. Assume that $s_{\varepsilon_1} \dots s_{\varepsilon_n} B_{X^*}$ is not empty. Then $s_{\varepsilon_1} \dots s_{\varepsilon_n}(MB_{|\ |^*})$ is non empty and by homogeneity, so is $s_{\frac{\varepsilon_1}{M}} \dots s_{\frac{\varepsilon_n}{M}}(B_{|\ |^*})$. Thus, if we denote σ_ε the Szlenk derivation on X^* where the diameter is taken with respect to the norm $|\ |$, we have that $\sigma_{\frac{\varepsilon_1}{M^2}} \dots \sigma_{\frac{\varepsilon_n}{M^2}} B_{|\ |^*}$ is non empty. Then classical manipulations on the Szlenk derivation imply that

$$\frac{1}{2}B_{|\ |^*} \subset \sigma_{\frac{\varepsilon_1}{4M^2}} \dots \sigma_{\frac{\varepsilon_n}{4M^2}} B_{|\ |^*} \subset \prod_{k=1}^{n}\big(1 - \frac{\gamma\varepsilon_k^q}{4^q M^{2q}}\big)B_{|\ |^*}.$$

The argument for the first inclusion can be found in [71] (proof of Proposition 3.3) and the second inclusion follows from the definition of $\overline{\theta}^*_{|\ |}$ and homogeneity. Finally we use the fact that $t \leq -\log(1-t)$, for $t \in [0,1)$ and elementary calculus to deduce that $\sum_{k=1}^n \varepsilon_k^q \leq \frac{4^q M^{2q}}{\gamma}\log 2$. This finishes the proof.

We now turn to (iv) implies (ii). This was already proved in [25] in a more general setting. We include the simpler proof in our situation for the sake of clarity. So, let M be such that if $\varepsilon_1, \dots, \varepsilon_n \geqslant 0$ are such that $s_{\varepsilon_1} \dots s_{\varepsilon_n}(B_{X^*}) \neq 0$, then $\sum_{i=1}^n \varepsilon_i^q \leqslant M^q$. Let D be a weak neighborhood base of 0 in X and assume that $c > 0$ is such that, for some $n \in \mathbb{N}$ and $(x_t)_{t\in D^{\leqslant n}}$ weakly null in B_X we have that for each $t \in D^n$, there exists $(a_i)_{i=1}^n \in B_{\ell_p^n}$ satisfying $\|\sum_{i=1}^n a_i x_{t|_i}\| > c$. For each $t \in D^n$, fix $x_t^* \in B_{X^*}$ and $(a_i^t)_{i=1}^n \in B_{\ell_p^n}$ such that

$$\text{Re } x_t^*\Big(\sum_{i=1}^n a_i^t x_{t|_i}\Big) = \Big\|\sum_{i=1}^n a_i^t x_{t|_i}\Big\| > c.$$

Define $f : D^n \to B_{\ell_\infty^n}$ by letting $f(t) = (x_t^*(x_{t|_1}), \dots, x_t^*(x_{t|_n}))$. Fix $\delta > 0$ arbitrary. By Corollary 1.2.8, there exist $(b_i)_{i=1}^n \in B_{\ell_\infty^n}$ and $\theta : D^n \to D^n$ preserving lengths and initial segments such that for all $t \in D^n$ and $1 \leqslant i \leqslant n$,

(i) if $\theta((U_1, \dots, U_k)) = (V_1, \dots, V_k)$, then $V_k \subset U_k$, and

(ii) $|x^*_{\theta(t)}(x_{\theta(t)|_i}) - b_i| < \delta$.

By replacing x_s with $x_{\theta(s)}$ and x_t^* with $x^*_{\theta(t)}$ for each $s \in D^{\leqslant n}$ and $t \in D^n$, we can relabel and assume that the original collections $(x_t)_{t\in D^{\leqslant n}}$ and $(x_t^*)_{t\in D^n} \subset B_{X^*}$ satisfy this property.

Define $\varepsilon_i = \max\{0, |b_i| - 2\delta\}$ for each $1 \leqslant i \leqslant n$. We will prove that for each $0 \leqslant j \leqslant n$ and $t \in D^{n-j}$, there exists $x^* \in s_{\varepsilon_{n-j+1}} \dots s_{\varepsilon_n}(B_{X^*})$ (that depends on t), and if $j < n$, this x^* can be chosen such that for each $1 \leqslant i \leqslant n-j$, $|x^*(x_{t|_i}) - b_i| \leqslant \delta$. We prove this claim by induction on j. By convention, in the $j=0$ case, $s_{\varepsilon_{n+1}}s_{\varepsilon_n}(B_{X^*})=B_{X^*}$ and we just take $x^*=x^*_t\in B_{X^*}$. Next, assume the result holds for some $0\leqslant j<n$. By the inductive hypothesis, for each $t\in D^{n-j-1}$ and $U\in D$, since $t\frown(U)\in D^{n-j}$, there exists $x^*_U\in s_{\varepsilon_{n-j+1}}\dots s_{\varepsilon_n}(B_{X^*})$ such that for each $1\leqslant i\leqslant n-j-1$, $|x^*_U(x_{t|_i})-b_i|\leqslant\delta$ and $|x^*_U(x|_{t\frown(U)})-b_{n-j}|\leqslant\delta$. If $\varepsilon_{n-j}=0$, we pick any U in D and set $x^*=x^*_U$. Note that the conclusions are satisfied, since, by convention

$$x^*_U\in s_{\varepsilon_{n-j+1}}\dots s_{\varepsilon_n}(B_{X^*})=s_{\varepsilon_{n-j}}s_{\varepsilon_{n-j+1}}\dots s_{\varepsilon_n}(B_{X^*}).$$

Consider now the case $\varepsilon_{n-j}>0$. If x^* is any weak*-cluster point of $(x^*_U)_{U\in D}$, then clearly $|x^*(x_{t|_i})-b_i|\leqslant\delta$ for each $1\leqslant i\leqslant n-j-1$. Note also that, since $(x_{t\frown(U)})_{U\in D}$ is weakly null, there exists $U_0\in D$ such that for all $U\subset U_0$, $|x^*(x_{t\frown(U)})|<\delta$. This implies that

$$\forall U\subset U_0,\quad \|x^*_U-x^*\|\geq|(x^*_U-x^*)(x_{t\frown(U)})|>|b_{n-j}|-2\delta=\varepsilon_{n-j}.$$

We now use that x^* is a weak*-cluster point of $(x^*_U)_{U\subset U_0}\subset s_{\varepsilon_{n-j+1}}\dots s_{\varepsilon_n}(B_{X^*})$ to deduce that $x^*\in s_{\varepsilon_{n-j}}\dots s_{\varepsilon_n}(B_{X^*})$ and $|x^*(x_{t|_i})-b_i|\leqslant\delta$. This finishes the inductive proof of our claim. Applying this claim for $j=n$ yields the existence of some $x^*\in s_{\varepsilon_1}\dots s_{\varepsilon_n}(B_{X^*})$, from which it follows that $\sum_{i=1}^n\varepsilon_i^q\leqslant M^q$. We can now use this information to estimate the constant c. We define $I=\{i\leqslant n:|b_i|>2\delta\}$. Then, for any $t\in D^n$,

$$\begin{aligned}c<\operatorname{Re}\,x^*_t\Big(\sum_{i=1}^n a_i^t x_{t|_i}\Big)&\leqslant\delta n+\sum_{i=1}^n|a_i^t||b_i|\leqslant3\delta n+\sum_{i\in I}|a_i^t||b_i|\\&\leqslant5\delta n+\sum_{j\in I}|a_i^t|\varepsilon_i\leqslant5\delta n+\|(a_i^t)_{i\in I}\|_{\ell_p^n}\|(\varepsilon_i)_{i\in I}\|_{\ell_q^n}\leqslant5\delta n+M.\end{aligned}$$

Since $\delta>0$ was arbitrary, we conclude that $c\leqslant M$. This finishes the proof of this last implication.

□

Next we describe the class N_p. A more general version of the following result is proved in [28].

Theorem 1.2.11. *Fix $1<p\leqslant\infty$ and let q be conjugate to p. Let X be a Banach space. The following are equivalent.*

(i) $X\in\mathsf{N}_p$.

(ii) There exists a constant $K>0$ such that for any $n\in\mathbb{N}$ and any weakly null collection $(x_t)_{t\in D^{\leq n}}$ in B_X, there exists $t\in D^n$ such that $\|\sum_{i=1}^n x_{t|_i}\|\leq Kn^{1/p}$.

(iii) There exists a constant $M\geq1$ and a constant $c>0$ such that for each $\sigma\in(0,1]$, there exists a norm $|\ |$ on X such that $M^{-1}|x|\leq\|x\|_X\leq M|x|$ for all $x\in X$ and

(a) if $1<p<\infty$, $\overline{\rho}_{|\ |}(\sigma)\leqslant c\sigma^p$

(b) if $p=\infty$, $\overline{\rho}_{|\ |}(c)\leqslant\sigma$.

(iv) There exists a constant $C>0$ such that $Cz(X,\varepsilon)\leq C\varepsilon^{-q}$, for all $\varepsilon\in(0,1)$.

Proof. Again, the equivalence between (i) and (ii) follows from our initial discussion on games.

The argument for $(ii) \Rightarrow (iii)$ is an adaptation of the proof of Theorem 4.2 in [46] to the non-separable case. We refer the reader to the part of this thesis coming from [30] or to [28].

Let us briefly explain the simple implication $(iii) \Rightarrow (ii)$. Let us assume, as we may, that $\| \ \|$ satisfies (iii) for $\sigma = \frac{1}{2}$. We shall show the existence of a constant $K \geq 2$ such that (ii) is satisfied. Let $(x_t)_{t\in D^{\leq n}}$ be a weakly null tree in B_X. Pick $t \in D^{\leq n}$ such that $2 < \|\sum_{i=1}^k x_{t|_i}\| \leq 3$ for some k (if this is not possible we are done). Now we pick recursively $U_{k+1}, \dots, U_n$ so that for all $k < l \leq n$, we have, if we denote $s = t \frown (U_{k+1}, \cdots U_n)$, $\|\sum_{i=1}^l x_{s|_i}\| > 2$ and $\|\sum_{i=1}^l x_{s|_i}\| \leq \|\sum_{i=1}^{l-1} x_{s|_i}\|(1 + 2c2^{-p})$. It now follows from classical use of Orlicz functions (see for instance the proof of Theorem 6.1 in [67]) that there exists a constant $K > 0$ so that $\|\sum_{i=1}^n x_{t|_i}\| \leq Kn^{1/p}$. □

We recall that P_p is defined to be $\bigcap_{1<r<p} \mathsf{T}_r$. Then we have.

Theorem 1.2.12. *Fix $1 < p \leqslant \infty$ and let q be conjugate to p. Let X be a Banach space. The following are equivalent*

(i) $X \in \mathsf{P}_p$.

(ii) *For each $1 < r < p$, X is r-AUS-able.*

(iii) *There exists an equivalent norm $|\cdot|$ on X such that for all $1 < r < p$, X is r-AUS.*

(iv) *For each $1 < r < p$, $\theta_n(X) = o(n^{1/r})$ (defined before Remark 1.2.3).*

(v) $\mathsf{p}(X) = \lim\limits_{\varepsilon\to 0^+} \frac{\log Sz(X,\varepsilon)}{|\log(\varepsilon)|} \leqslant q$.

Proof. The equivalence between (i) and (ii) follows from Theorem 1.2.9. The fact that (ii) implies (v) follows from Proposition 1.2.2 and the existence of a universal constant $C \geq 1$ such that

$$\forall \varepsilon \in (0,1),\ Sz(X,\varepsilon) \leq C\left(\overline{\delta}_X^*\left(\frac{\varepsilon}{C}\right)\right)^{-1}.$$

The implication $(v) \Rightarrow (iii)$ is proved in [27] in a very general setting (non-separable, for higher ordinals and operators). Obviously (iii) implies (ii). The implication $(ii) \Rightarrow (iv)$ also follows from Theorem 1.2.9. Finally $(iv) \Rightarrow (ii)$ relies on an averaging of the norms provided by (iii) in Theorem 1.2.11. □

We finally summarize what is known about the inclusions between these classes.

Theorem 1.2.13. *Recall that D_1 denotes the class the of all Banach spaces with Szlenk index at most ω. Then*

(i) $\mathsf{D}_1 = \bigcup_{1<p\leqslant\infty} \mathsf{T}_p = \bigcup_{1<p\leqslant\infty} \mathsf{A}_p = \bigcup_{1<p\leqslant\infty} \mathsf{N}_p = \bigcup_{1<p\leqslant\infty} \mathsf{P}_p$.

(ii) *For $1 < p < \infty$, $\mathsf{T}_p \subsetneq \mathsf{A}_p \subsetneq \mathsf{N}_p \subsetneq \mathsf{P}_p$.*

(iii) $\mathsf{T}_\infty \subsetneq \mathsf{A}_\infty = \mathsf{N}_\infty \subsetneq \mathsf{P}_\infty$.

Proof. Let $1 < p \leq \infty$. We clearly have that $\mathsf{T}_p \subset \mathsf{A}_p \subset \mathsf{N}_p \subset \mathsf{P}_p$. It follows from (iii) in Theorem 1.2.9 and (ii) in Theorem 1.2.12 that $\mathsf{P}_p \subset \mathsf{D}_1$. We have already explained that if $X \in \mathsf{D}_1$, then $\mathsf{p}(X) < \infty$, so Theorem 1.2.12 implies that $X \in \mathsf{T}_r$, for some $1 < r < p$. Our statement (i) follows from gathering all these pieces of information.

The fact that the inclusions are strict in (ii), as well as $\mathsf{T}_\infty \neq \mathsf{A}_\infty = \mathsf{N}_\infty \neq \mathsf{P}_\infty$ are proved in [28].

□

1.2.3 Separable determination

We start with a simple but fundamental statement about selecting weakly null sequences from weakly null nets in AUS-able Banach spaces.

Proposition 1.2.14. *Let X be a Banach space with $Sz(X) \leqslant \omega$. Let D be a weak neighborhood base at 0 in X. For any $(x_U)_{U\in D} \subset B_X$ such that $x_U \in U$ for all $U \in D$, there exists a function $f : \mathbb{N} \to D$ such that $(x_{f(n)})_{n=1}^\infty$ is a weakly null sequence.*

Proof. Since $Sz(X) \leqslant \omega$, $X \in \mathsf{T}_r$ for some $1 < r < \infty$. Let $1/r + 1/s = 1$ and $c > \mathsf{t}_r(X)$. Let ϕ be a winning strategy for Player I in the $T(c, r)$ game. Let V_1 be determined by ϕ and fix $U_1 \in D$ such that $U_1 \subset V_1$. Let Player II choose $x_{U_1} \in U_1 \cap B_X$. Let V_2 be determined by ϕ and fix $U_2 \in D$ such that $U_2 \subset V_2$. Let Player II choose $x_{U_2} \in U_2 \cap B_X$. Continue in this way until $U_1, U_2, \ldots$ have been chosen. Define $f(n) = U_n$ and note that $\|(x_{f(n)})_{n=1}^\infty\|_s^w = \|(x_{U_n})_{n=1}^\infty\|_s^w \leqslant c < \infty$. Therefore $(x_{f(n)})_{n=1}^\infty$ is weakly null. □

We are now ready to give a unified proof of the separable determination of all the properties considered in this chapter. Before we state it, let us mention that summable Szlenk index and having power type Szlenk index was proved to be separably determined by Draga and Kochanek in [36].

Theorem 1.2.15. *If X is a Banach space with $Sz(X) \leqslant \omega$, then for each $1 < p \leqslant \infty$,*

$$\mathsf{t}_p(X) = \sup\{\mathsf{t}_p(E) : E \leqslant X \text{ is separable}\},$$

and this supremum is attained, although possibly infinite. The same is true of $\mathsf{a}_p(X)$, $\mathsf{n}_p(X)$, and $\theta_n(X)$. In particular, if X is a Banach space all of whose separable subspaces lie in T_p, then X lies in T_p. The same conclusion holds for A_p, N_p, P_p and D_1.

Proof. It is clear that $\mathsf{t}_p(X) \geqslant \sup\{\mathsf{t}_p(E) : E \leqslant X \text{ is separable}\}$. If $c < \mathsf{t}_p(X)$, then there exists a weakly null collection $(x_t)_{t\in D^{<\omega}}$ such that for each $\tau \in D^\omega$, $\|(x_{\tau|_i})_{i=1}^\infty\|_q^w > c$.

First, we build $\varphi : \mathbb{N}^{<\omega} \to D^{<\omega}$ which preserves lengths and immediate predecessors such that $(x_{\varphi(t)})_{t\in\mathbb{N}^{<\omega}}$ is weakly null. We define $\varphi(t)$ by induction on $|t|$. By Proposition 1.2.14 applied to $(x_{(U)})_{U\in D}$, there exists $f : \mathbb{N} \to D$ such that $(x_{(f(n))})_{n=1}^\infty$ is weakly null. Define $\varphi((n)) = (f(n))$. Next, if $\varphi(t)$ has been defined, apply Proposition 1.2.14 to $(x_{\varphi(t)\frown(U)})_{U\in D}$ to select $g : \mathbb{N} \to D$ such that $(x_{\varphi(t)\frown(g(n))})_{n=1}^\infty$ is weakly null. Define $\varphi(t \frown (n)) = \varphi(t) \frown (g(n))$. This completes the construction.

Define $y_t = x_{\varphi(t)}$. It follows that for any $\tau_1 \in \mathbb{N}^\omega$, there exists a unique $\tau \in D^\omega$ such that $\varphi(\tau_1|_i) = \tau|_i$ for all $i \in \mathbb{N}$, so that

$$\|(y_{\tau_1|_i})_{i=1}^\infty\|_q^w = \|(x_{\tau|_i})_{i=1}^\infty\|_q^w > c.$$

Therefore if F is the closed linear span of $(y_t)_{t\in\mathbb{N}^{<\omega}}$, then $\mathsf{t}_p(F) > c$. This shows that $\mathsf{t}_p(X) \leqslant \sup\{\mathsf{t}_p(E) : E \leqslant X \text{ is separable}\}$. Next, let R denote the set of rational numbers r such that $\mathsf{t}_p(X) > r$. For each $r \in R$, let F_r be a separable subspace of X such that $\mathsf{t}_p(F_r) > r$, and let E be the closed span of E_r, $r \in R$. Then $\mathsf{t}_p(E) = \mathsf{t}_p(X)$, and the supremum is attained. The arguments for $\mathsf{a}_p(X), \mathsf{n}_p(X), \theta_n(X)$ are similar.

If X is a Banach space all of whose separable subspaces lie in $\mathsf{T}_p \subset \mathsf{D}_1$, then $\mathsf{t}_p(X) = \sup\{\mathsf{t}_p(E) : E \leqslant X \text{ is separable}\}$ must be finite. Indeed, if the supremum were infinite, then since it is attained, there would exist some separable $E \leqslant X$ such that $\mathsf{t}_p(E) = \infty$, and E does not belong to T_p. Similar arguments hold for A_p and N_p.

For P_p, we note that

$$\begin{aligned} X \in \mathsf{P}_p &\Leftrightarrow (\forall 1 < r < p)(X \in \mathsf{T}_r) \\ &\Leftrightarrow (\forall 1 < r < p)(\forall E \leqslant X \text{ separable})(E \in \mathsf{T}_r) \\ &\Leftrightarrow (\forall E \leqslant X \text{ separable})(\forall 1 < r < p)(E \in \mathsf{T}_r) \\ &\Leftrightarrow (\forall E \leqslant X \text{ separable})(E \in \mathsf{P}_p) \end{aligned}$$

Assume now that X is not in D_1. Then, for any $p \in \mathbb{Q} \cap (1, \infty)$, X does not belong to T_p. So for any $p \in \mathbb{Q} \cap (1, \infty)$, there exists a separable subspace E_p of X so that E_p is not in T_p. Then the closed linear span of these E_p's is a separable subspace of X which does not belong to D_1.

□

Chapter 2

Hamming graphs and concentration properties in non-quasi-reflexive Banach spaces

In this chapter, we study some concentration properties for Lipschitz maps defined on Hamming graphs with values in Banach spaces. We investigate their stability under some general sums of Banach spaces, including ℓ_p-sums. As an application, we extend a result of Causey on the coarse Lipschitz structure of quasi-reflexive spaces satisfying upper ℓ_p tree estimates to the setting of ℓ_p-sums of such spaces. Our result provides us with a tool for constructing the first examples of Banach spaces that are not quasi-reflexive but nevertheless admit some concentration inequality. We also give a sufficient condition for a space to be asymptotic-c_0 in terms of a concentration property, as well as relevant counterexamples.

Let us briefly describe the content of this chapter. We start by defining the Hamming graphs and the concentration properties that will be studied. In a second section, we prove our main result, Theorem B, in order to exhibit the first non-quasi-reflexive space that cannot equi-Lipschitz contain the Hamming graphs and we raise a few questions. Finally, we prove that a Banach space that has the concentration property HC_∞ is asymptotic-c_0 and we use a result of Schlumprecht to deduce the existence of a separable asymptotic-c_0 dual space that does not have any of the concentration properties we will introduce. This chapter is based on [40].

2.1 Definitions and notation

2.1.1 Hamming graphs

Before introducing the concentration properties, we need to define special metric graphs that we shall call *Hamming graphs*. Let $\mathbb{M}$ be an infinite subset of $\mathbb{N}$. We denote by $[\mathbb{M}]^\omega$ the set of infinite subsets of $\mathbb{M}$. For $\mathbb{M} \in [\mathbb{N}]^\omega$ and $k \in \mathbb{N}$, let

$$[\mathbb{M}]^k = \{\overline{n} = (n_1, \dots, n_k) \in \mathbb{M}^k; n_1 < \dots < n_k\},$$

$$[\mathbb{M}]^{\leq k} = \bigcup_{j=1}^{k} [\mathbb{M}]^j \cup \{\varnothing\},$$

and

$$[\mathbb{M}]^{<\omega} = \bigcup_{k=1}^{\infty} [\mathbb{M}]^k \cup \{\varnothing\}.$$

Let us point out the fact that we will identify some $\mathbb{M} \in [\mathbb{M}]^{<\omega} \cup [\mathbb{M}]^{\omega}$ to its increasing enumeration.
Then we equip $[\mathbb{M}]^k$ with the *Hamming distance*:

$$d_{\mathbb{H}}(\overline{n}, \overline{m}) = |\{j; n_j \neq m_j\}|$$

for all $\overline{n} = (n_1, \ldots, n_k), \overline{m} = (m_1, \ldots, m_k) \in [\mathbb{M}]^k$.
Let us mention that this distance can be extended to $[\mathbb{M}]^{<\omega}$ by letting

$$d_{\mathbb{H}}(\overline{n}, \overline{m}) = |\{i \in \{1, \cdots, \min(l, j)\}; n_i \neq m_i\}| + \max(l, j) - \min(l, j)$$

for all $\overline{n} = (n_1, \ldots, n_l), \overline{m} = (m_1, \ldots, m_j) \in [\mathbb{M}]^{<\omega}$ (with possibly $l = 0$ or $j = 0$).
We also need to introduce $I_k(\mathbb{M})$, the set of strictly interlaced pairs in $[\mathbb{M}]^k$:

$$I_k(\mathbb{M}) = \{(\overline{n}, \overline{m}) \subset [\mathbb{M}]^k; n_1 < m_1 < \cdots < n_k < m_k\}$$

and, for each $j \in \{1, \cdots, k\}$, let

$$H_j(\mathbb{M}) = \{(\overline{n}, \overline{m}) \subset [\mathbb{M}]^k; \forall i \neq j, n_i = m_i \text{ and } n_j < m_j\}.$$

Note that, for $(\overline{n}, \overline{m}) \in I_k(\mathbb{M})$, $d_{\mathbb{H}}(\overline{n}, \overline{m}) = k$ and $\overline{n} \cap \overline{m} = \varnothing$.

Let us mention that, in this chapter, we will only be interested in the Hamming distance but originally, when Hamming graphs were used in [67], it could be replaced (except for their last Theorem 6.1) by the *symmetric distance*, defined by

$$d_{\Delta}(\overline{n}, \overline{m}) = \frac{1}{2}|\overline{n} \triangle \overline{m}|$$

for all $\overline{n}, \overline{m} \in [\mathbb{N}]^{<\omega}$, where $\overline{n} \triangle \overline{m}$ denotes the symmetric difference between $\overline{n}$ and $\overline{m}$.

2.1.2 Definitions of concentration properties

In this subsection, we introduce all the concentration properties mentioned in this chapter. Before doing so, let us recall a version of Ramsey's Theorem we will use several times.

Theorem 2.1.1 (Ramsey's Theorem [94]). *Let $k \in \mathbb{N}$ and $\mathcal{A} \subset [\mathbb{N}]^k$. There exists $\mathbb{M} \in [\mathbb{N}]^{\omega}$ such that either $[\mathbb{M}]^k \subset \mathcal{A}$ or $[\mathbb{M}]^k \cap \mathcal{A} = \varnothing$.*

The following properties are studied in [67], [74] and [11]. We use the convention $1/\infty = 0$.

Definition 2.1.2. Let (X, d) be a metric space, $\lambda > 0$, $p \in (1, \infty]$.
• We say that X has property λ-*HFC$_p$* (Hamming Full Concentration) if, for all $k \in \mathbb{N}$, for every Lipschitz function $f : ([\mathbb{N}]^k, d_{\mathbb{H}}) \to X$, one can find $\mathbb{M} \in [\mathbb{N}]^{\omega}$ such that

$$\forall \overline{n}, \overline{m} \in [\mathbb{M}]^k, \;\; d(f(\overline{n}), f(\overline{m})) \leq \lambda k^{\frac{1}{p}} \mathrm{Lip}(f).$$

We say that X has property *HFC_p* if X has property λ-HFC$_p$ for some $\lambda > 0$.
• We say that X has property *λ-HIC_p* (Hamming Interlaced Concentration) if, for all $k \in \mathbb{N}$, for every Lipschitz function $f : ([\mathbb{N}]^k, d_{\mathbb{H}}) \to X$, one can find $(\overline{n}, \overline{m}) \in I_k(\mathbb{N})$ satisfying

$$d(f(\overline{n}), f(\overline{m})) \leq \lambda k^{\frac{1}{p}} \mathrm{Lip}(f).$$

We say that X has property *HIC_p* if X has property λ-HIC$_p$, for some $\lambda > 0$.

Remark 2.1.3. 1) Let us notice that, by Ramsey's Theorem ($I_k(\mathbb{N})$ can be identified with $[\mathbb{N}]^{2k}$), a metric space (X, d) has property λ-HIC$_p$ if and only if, for all $k \in \mathbb{N}$, for every Lipschitz function $f : ([\mathbb{N}]^k, d_{\mathbb{H}}) \to X$, one can find $\mathbb{M} \in [\mathbb{N}]^\omega$ that satisfies

$$\forall (\overline{n}, \overline{m}) \in I_k(\mathbb{M}), \;\; d(f(\overline{n}), f(\overline{m})) \leq \lambda k^{\frac{1}{p}} \mathrm{Lip}(f).$$

2) Baudier, Lancien, Motakis and Schlumprecht showed that property HFC$_\infty$ is equivalent for a Banach space to being reflexive and asymptotic-c_0, *i.e* in A_∞ (see [11] for the proof of this result and Section 2.3 or Chapter 1 for a reminder of the definition of asymptotic-c_0).

We now introduce a property that seems weaker than the previous one but is enough to prevent the equi-Lipschitz embedding (or equi-coarse embedding for the case $p = \infty$) of Hamming graphs. We will show later that this property actually coincides with property HIC$_p$, $p \in (1, \infty]$.

Definition 2.1.4. Let (X, d) be a metric space, $\lambda > 0$, and $p \in (1, \infty]$. We say that X has property *λ-HC_p* if, for all $k \in \mathbb{N}$, for every Lipschitz function $f : ([\mathbb{N}]^k, d_{\mathbb{H}}) \to X$, one can find $\overline{n}, \overline{m} \in [\mathbb{N}]^k$ satisfying $\overline{n} \cap \overline{m} = \varnothing$ and

$$d(f(\overline{n}), f(\overline{m})) \leq \lambda k^{\frac{1}{p}} \mathrm{Lip}(f).$$

We say that X has property *HC_p* if X has property λ-HC$_p$, for some $\lambda > 0$.

One can check that all these concentration properties are stable under coarse Lipschitz embeddings and that properties HFC$_\infty$, HC$_\infty$ and HIC$_\infty$ are even stable under coarse embeddings, when the embedded space is a Banach space (the key points can be found in the proof of Proposition 2.1.8).

Let us now introduce the last concentration properties we will study here, more precise than HC$_p$ and HIC$_p$, $p \in (1, \infty)$, where directional Lipschitz constants take part, hence the "d" in subscript in the acronyms below.

Definition 2.1.5. Let (X, d) be a metric space, $\lambda > 0$, $p \in (1, \infty)$.
• We say that X has property *λ-$HFC_{p,d}$* (resp. *λ-$HIC_{p,d}$*) if, for every $k \in \mathbb{N}$ and every Lipschitz function $f : ([\mathbb{N}]^k, d_{\mathbb{H}}) \to X$, there exists $\mathbb{M} \in [\mathbb{N}]^\omega$ such that

$$d(f(\overline{n}), f(\overline{m})) \leq \lambda \left(\sum_{j=1}^{k} \alpha_j^p \right)^{\frac{1}{p}}$$

for all $\overline{n}, \overline{m} \in [\mathbb{M}]^k$ (resp. $(\overline{n}, \overline{m}) \in I_k(\mathbb{M})$), where, for each $j \in \{1, \cdots, k\}$

$$\alpha_j = \sup_{(\overline{n}, \overline{m}) \in H_j(\mathbb{N})} d(f(\overline{n}), f(\overline{m}))$$

is the *j-th directional Lipschitz constant of f.*
We say that X has property *$HFC_{p,d}$* (resp. *$HIC_{p,d}$*) if X has property λ-HFC$_{p,d}$ (resp. λ-HIC$_{p,d}$), for some $\lambda > 0$.
• Similary, we say that X has property *λ-$HC_{p,d}$* if, for every $k \in \mathbb{N}$ and every Lipschitz function $f : ([\mathbb{N}]^k, d_{\mathbb{H}}) \to X$, one can find $\overline{n}, \overline{m} \in [\mathbb{N}]^k$ satisfying $\overline{n} \cap \overline{m} = \varnothing$ and

$$d(f(\overline{n}), f(\overline{m})) \leq \lambda \left(\sum_{j=1}^{k} \alpha_j^p \right)^{\frac{1}{p}}$$

where the α_j, $j \in \{1, \cdots, k\}$, are defined as above.
We say that X has property *$HC_{p,d}$* if X has property λ-HC$_{p,d}$, for some $\lambda > 0$.

Remark 2.1.6. Let $f : ([\mathbb{N}]^k, d_{\mathbb{H}}) \to X$ be a Lipschitz map, $(\alpha_j)_{j=1}^k$ its directional Lipschitz constants. We have

$$\mathrm{Lip}(f) = \max_{1 \leq j \leq k} \alpha_j.$$

Indeed, the inequality $\mathrm{Lip}(f) \geq \max_{1 \leq j \leq k} \alpha_j$ is clear and if $\overline{n}$, $\overline{m}$ are such that $d(f(\overline{n}), f(\overline{m})) = \mathrm{Lip}(f) d_{\mathbb{H}}(\overline{n}, \overline{m})$ with $\ell = d_{\mathbb{H}}(\overline{n}, \overline{m}) \geq 2$, then one can find $\overline{s}_1, \ldots, \overline{s}_{\ell-1} \in [\mathbb{N}]^k$ so that

$$d_{\mathbb{H}}(\overline{n}, \overline{s}_1) = 1, d_{\mathbb{H}}(\overline{s}_{\ell-1}, \overline{m}) = 1, \text{ and } d_{\mathbb{H}}(\overline{s}_i, \overline{s}_{i+1}) = 1 \text{ for every } 1 \leq i \leq \ell - 2.$$

Thus

$$\begin{aligned} \mathrm{Lip}(f) d_{\mathbb{H}}(\overline{n}, \overline{m}) &= d(f(\overline{n}), f(\overline{m})) \\ &\leq d(f(\overline{n}), f(\overline{s}_1)) + \sum_{i=1}^{\ell-2} d(f(\overline{s}_i), f(\overline{s}_{i+1})) + d(f(\overline{s}_{\ell-1}), f(\overline{m})) \\ &\leq \max_{1 \leq j \leq k} \alpha_j + \sum_{i=1}^{\ell-2} \max_{1 \leq j \leq k} \alpha_j + \max_{1 \leq j \leq k} \alpha_j \\ &= \ell \max_{1 \leq j \leq k} \alpha_j = (\max_{1 \leq j \leq k} \alpha_j) d_{\mathbb{H}}(\overline{n}, \overline{m}) \end{aligned}$$

Remark 2.1.7. It is important to note that Theorem 6.1 [67] and Theorem 5.2 [28] can be rephrased as follows: for $p \in (1, \infty)$, a reflexive (resp. quasi-reflexive) Banach space belonging to A_p has property HFC$_{p,d}$ (resp. HC$_{p,d}$) and a reflexive (resp. quasi-reflexive) Banach space belonging to N_p has property HFC$_p$ (resp. HC$_p$). Even though Kalton and Randrianarivony [67] proved their theorem for reflexive p-AUS Banach spaces, their proof implicitly contains the latter result. Let us also note that a Banach space with property HFC$_p$ is necessarily reflexive (see [8]). In 2017, Lancien and Raja [74] proved that all quasi-reflexive p-AUS Banach spaces have property HC$_{p,d}$. It was later extended as mentioned by Causey [28].

The stability of these last properties under coarse Lipschitz embeddings when the embedded space is a Banach space is a bit less clear than for the non-directional case so we include a proof for completeness.

Proposition 2.1.8. *Let $p \in (1, \infty)$, $P \in \{HFC_{p,d}, HIC_{p,d}, HC_{p,d}\}$, X a Banach space and (Y, d_Y) a metric space.*
If Y has property P and X coarse Lipschitz embeds into Y, then X has property P.

Proof. We only prove the stability of $\mathrm{HFC}_{p,d}$, the proofs for the other two properties are similar.
Let us assume that Y has property λ-$\mathrm{HFC}_{p,d}$ for a $\lambda > 0$ and that there exist a map $\varphi : X \to Y$ and $A, B, C, D > 0$ such that $\rho_\varphi(t) \geq At - B$ and $\omega_\varphi(t) \leq Ct + D$ for all $t \geq 0$.
Let $k \in \mathbb{N}$, $f : ([\mathbb{N}]^k, d_{\mathbb{H}}) \to X$ a Lipschitz function with $\mathrm{Lip}(f) > 0$.
Without loss of generality, we can assume that, for all $j \in \{1, \cdots, k\}$, we have

$$\alpha_j = \sup_{(\overline{n},\overline{m}) \in H_j(\mathbb{M})} \|f(\overline{n}) - f(\overline{m})\| > 0.$$

Indeed, by Remark 2.1.6, $\max_{j\in\{1,\cdots,k\}} \alpha_j = \mathrm{Lip}(f) > 0$ and if $\alpha_j = 0$ for some $j \in \{1, \cdots, k\}$, then the expression of f does not depend on this j^{th} coordinate.
Therefore

$$\alpha = \min_{1\leq j\leq k} \alpha_j \in (0, \mathrm{Lip}(f)].$$

Let us note that $\omega_\varphi(t) \leq (C + D)t$ for all $t \geq 1$ so, for all $j \in \{1, \cdots, k\}$ and for all $(\overline{n}, \overline{m}) \in H_j(\mathbb{N})$, we have

$$d_Y\left(\varphi\left(\frac{1}{\alpha}f(\overline{n})\right), \varphi\left(\frac{1}{\alpha}f(\overline{m})\right)\right) \leq \omega_\varphi\left(\frac{\alpha_j}{\alpha}\right) \leq \frac{C + D}{\alpha}\alpha_j.$$

Now, by assumption on Y, we can find $\mathbb{M} \in [\mathbb{N}]^\omega$ so that

$$d_Y\left(\varphi\left(\frac{1}{\alpha}f(\overline{n})\right), \varphi\left(\frac{1}{\alpha}f(\overline{m})\right)\right) \leq \frac{\lambda(C + D)}{\alpha}\left(\sum_{j=1}^k \alpha_j^p\right)^{\frac{1}{p}}$$

for all $\overline{n}, \overline{m} \in [\mathbb{M}]^k$.
Thus

$$\|f(\overline{n}) - f(\overline{m})\| \leq \frac{\lambda(C + D)}{A}\left(\sum_{j=1}^k \alpha_j^p\right)^{\frac{1}{p}} + \frac{\alpha B}{A} \leq \frac{\lambda(C + D) + B}{A}\left(\sum_{j=1}^k \alpha_j^p\right)^{\frac{1}{p}}$$

for all $\overline{n}, \overline{m} \in [\mathbb{M}]^k$. Consequently, X has property $\mathrm{HFC}_{p,d}$. □

As promised, the next proposition shows that properties HC_p and HIC_p, $p \in (1, \infty]$, are equivalent. This explains why we will only talk about property HC_∞ in the last section.
Before proving this result, let us introduce some vocabulary. Let $\mathbb{M} \in [\mathbb{N}]^\omega$. For $\overline{n}, \overline{m} \in [\mathbb{M}]^k$ satisfying $\overline{n} \cap \overline{m} = \varnothing$, we denote by ϕ the unique increasing bijection from $\overline{n} \cup \overline{m}$ onto $\{1, \cdots, 2k\}$. If

$$I = \{A \subset \{1, \cdots, 2k\}; |A| = k\},$$

we say that $(\overline{n}, \overline{m})$ is in position $A \in I$ if $\phi(\overline{n}) = A$.
Thus, we note that the pair $(\overline{n}, \overline{m})$ with $\overline{n}, \overline{m} \in [\mathbb{M}]^k$ and $\overline{n} \cap \overline{m} = \varnothing$, can be in $\binom{2k-1}{k-1}$ possible different positions if we ask 1 to be in the position (and we can do it without loss of generality). We denote these positions by $\mathcal{P}_i^k(\mathbb{M})$, $i \in \left\{1, \cdots, \binom{2k-1}{k-1}\right\}$. Let us remark that, once a position is fixed, each pair $(\overline{n}, \overline{m})$ in this position can be identified with an element of $[\mathbb{M}]^{2k}$, which will allow us to use Ramsey's Theorem.

Proposition 2.1.9. *For every $p \in (1, \infty]$, properties HC_p and HIC_p are equivalent. More precisely, a metric space with property λ-HIC_p, for some $\lambda > 0$, has property λ-HC_p and a metric space with property λ-HC_p has property 2λ-HIC_p.*

Proof. For every $p \in (1, \infty]$, $\lambda > 0$, the implication λ-$\mathrm{HIC}_p \implies \lambda$-$\mathrm{HC}_p$ is clear so let us show the other implication.
We will do it with $p = \infty$, the other cases can be treated similarly.
Let (X, d) be a metric space with property λ-HC_∞ for some $\lambda > 0$. Let $k \in \mathbb{N}$, $f : ([\mathbb{N}]^k, d_{\mathbb{H}}) \to X$ a Lipschitz function.
For each $\mathbb{M} \in [\mathbb{N}]^\omega$, there exist $i \in \left\{1, \cdots, \binom{2k-1}{k-1}\right\}$ and $(\overline{n}, \overline{m}) \in \mathcal{P}_i^k(\mathbb{M})$ such that $d(f(\overline{n}), f(\overline{m})) \leq \lambda \mathrm{Lip}(f)$.
Let us show that there exist $i \in \left\{1, \cdots, \binom{2k-1}{k-1}\right\}$ and $\mathbb{M} \in [\mathbb{N}]^\omega$ such that $d(f(\overline{n}), f(\overline{m})) \leq \lambda \mathrm{Lip}(f)$ for all $(\overline{n}, \overline{m}) \in \mathcal{P}_i^k(\mathbb{M})$.
By Ramsey's Theorem, if $\mathcal{A}_1 = \{(\overline{n}, \overline{m}) \in \mathcal{P}_1^k(\mathbb{N}); d(f(\overline{n}), f(\overline{m})) \leq \lambda \mathrm{Lip}(f)\} \subset \mathcal{P}_1^k(\mathbb{N})$, there exists $\mathbb{M}_1 \in [\mathbb{N}]^\omega$ such that $\mathcal{P}_1^k(\mathbb{M}_1) \subset \mathcal{A}_1$ or $\mathcal{P}_1^k(\mathbb{M}_1) \cap \mathcal{A}_1 = \varnothing$.
If $\mathcal{P}_1^k(\mathbb{M}_1) \cap \mathcal{A}_1 = \varnothing$, we apply the same result with $\mathcal{A}_2 = \{(\overline{n}, \overline{m}) \in \mathcal{P}_2^k(\mathbb{M}_1); d(f(\overline{n}), f(\overline{m})) \leq \lambda \mathrm{Lip}(f)\} \subset \mathcal{P}_2^k(\mathbb{M}_1)$ and we get $\mathbb{M}_2 \in [\mathbb{M}_1]^\omega$ such that $\mathcal{P}_2^k(\mathbb{M}_2) \subset \mathcal{A}_2$ or $\mathcal{P}_2^k(\mathbb{M}_2) \cap \mathcal{A}_2 = \varnothing$.
We continue this way inductively.
As X has property λ-HC_∞, we cannot repeat this operation for all $\binom{2k-1}{k-1}$ positions so there exist $i \in \left\{1, \cdots, \binom{2k-1}{k-1}\right\}$ and $\mathbb{M} \in [\mathbb{N}]^\omega$ such that $d(f(\overline{n}), f(\overline{m})) \leq \lambda \mathrm{Lip}(f)$ for all $(\overline{n}, \overline{m}) \in \mathcal{P}_i^k(\mathbb{M})$.
Let us show that there exists $(\overline{n}, \overline{m}) \in I_k(\mathbb{N})$ such that $d(f(\overline{n}), f(\overline{m})) \leq 2\lambda \mathrm{Lip}(f)$.
For that, let $\mathbb{M} = \{q_1 < q_2 < \cdots < q_j < \cdots\}$.
Now, we just have to observe that we can choose $(\overline{n}, \overline{p}) \in \mathcal{P}_i^k(\mathbb{M})$ such that $n_1 < p_1$ and $\overline{n}, \overline{p} \subset \{q_1, q_{2k+1}, \cdots, q_{2k(2k-1)+1}\}$. This leaves us enough space to get an element $\overline{m} \in [\mathbb{M}]^k$ so that $(\overline{n}, \overline{m}) \in I_k(\mathbb{M})$ and $(\overline{m}, \overline{p}) \in \mathcal{P}_i^k(\mathbb{M})$.
The result follows from the triangle inequality. □

Remark 2.1.10. With a similar proof, we can prove that properties $\mathrm{HC}_{p,d}$ and $\mathrm{HIC}_{p,d}$ are equivalent.

2.2 Stability under sums

2.2.1 Statements

In order to prove the stability of property $\mathrm{HC}_{p,d}$, $p \in (1, \infty)$, under ℓ_p sums, the idea is to adapt Braga's proof of Proposition 7.2 in [19] with property $\mathrm{HIC}_{p,d}$ instead of property p-Banach-Saks.
To do so, we need the following proposition. We chose to state it with property $\mathrm{HIC}_{p,d}$, which we recall is equivalent to property $\mathrm{HC}_{p,d}$, but the same result can be proved for property $\mathrm{HFC}_{p,d}$ with a similar proof.

Proposition 2.2.1. *Let $p \in (1, +\infty)$, $\lambda > 0$, E be a Banach space with a normalized 1-unconditional p-convex basis $(e_n)_{n\in\mathbb{N}}$ with convexity constant 1.*
For every $n \in \mathbb{N}$ and every finite sequence $(X_j)_{j=1}^n$ of Banach spaces having property λ-$HIC_{p,d}$, the space $\left(\sum_{j=1}^n X_j\right)_E$ has property $(\lambda + \varepsilon)$-$HIC_{p,d}$ for any $\varepsilon > 0$.

Proof. It is enough to prove this result for $X = X_1 \bigoplus_E X_2$.
Let $k \in \mathbb{N}$, $\mathbb{M} \in [\mathbb{N}]^\omega$, $\varepsilon > 0$, $h = (f,g) : ([\mathbb{M}]^k, d_{\mathbb{H}}) \to X$ a Lipschitz function.
For each $j \in \{1, \cdots, k\}$, let $\gamma_j = \sup_{(\overline{n},\overline{m}) \in H_j(\mathbb{M})} \|h(\overline{n}) - h(\overline{m})\|$.
There exists $\varepsilon' > 0$ such that

$$\lambda^p \sum_{j=1}^k (\gamma_j + 2\varepsilon')^p \leq (\lambda + \varepsilon)^p \sum_{j=1}^k \gamma_j^p.$$

Let $\alpha_1 = \inf_{\mathbb{M}_1 \in [\mathbb{M}]^\omega} \sup_{(\overline{n},\overline{m}) \in H_1(\mathbb{M}_1)} \|f(\overline{n}) - f(\overline{m})\|$.
There exists $\mathbb{M}_1 \in [\mathbb{M}]^\omega$ so that $\|f(\overline{n}) - f(\overline{m})\| \leq \alpha_1 + \varepsilon'$ for every $(\overline{n},\overline{m}) \in H_1(\mathbb{M}_1)$.
Let $\beta_1 = \inf_{\mathbb{M}'_1 \in [\mathbb{M}_1]^\omega} \sup_{(\overline{n},\overline{m}) \in H_1(\mathbb{M}'_1)} \|g(\overline{n}) - g(\overline{m})\|$.
There exists $\mathbb{M}'_1 \in [\mathbb{M}_1]^\omega$ so that $\|g(\overline{n}) - g(\overline{m})\| \leq \beta_1 + \varepsilon'$ for every $(\overline{n},\overline{m}) \in H_1(\mathbb{M}'_1)$.
We continue inductively this way until we define α_k and β_k as follows.
Let $\alpha_k = \inf_{\mathbb{M}_k \in [\mathbb{M}'_{k-1}]^\omega} \sup_{(\overline{n},\overline{m}) \in H_k(\mathbb{M}_k)} \|f(\overline{n}) - f(\overline{m})\|$.
There exists $\mathbb{M}_k \in [\mathbb{M}'_{k-1}]^\omega$ so that $\|f(\overline{n}) - f(\overline{m})\| \leq \alpha_k + \varepsilon'$ for every $(\overline{n},\overline{m}) \in H_k(\mathbb{M}_k)$.
Let $\beta_k = \inf_{\mathbb{M}'_k \in [\mathbb{M}_k]^\omega} \sup_{(\overline{n},\overline{m}) \in H_k(\mathbb{M}'_k)} \|g(\overline{n}) - g(\overline{m})\|$.
There exists $\mathbb{M}'_k \in [\mathbb{M}_k]^\omega$ so that $\|g(\overline{n}) - g(\overline{m})\| \leq \beta_k + \varepsilon'$ for every $(\overline{n},\overline{m}) \in H_k(\mathbb{M}'_k)$.
• Let us begin by showing that $\|\alpha_j e_1 + \beta_j e_2\| \leq \gamma_j$ for all $j \in \{1, \cdots, k\}$.
For that, assume that there exists $j \in \{1, \cdots, k\}$ such that $\|\alpha_j e_1 + \beta_j e_2\| > \gamma_j$. Then, there exists $\eta > 0$ so that $\|(\alpha_j - \eta)e_1 + (\beta_j - \eta)e_2\| > \gamma_j$ by 1-unconditionality.
$*$ If there exists $(\overline{n},\overline{m}) \in H_j(\mathbb{M}'_k)$ such that $\|f(\overline{n}) - f(\overline{m})\| \geq \alpha_j - \eta$ and $\|g(\overline{n}) - g(\overline{m})\| \geq \beta_j - \eta$, then $\|h(\overline{n}) - h(\overline{m})\| > \gamma_j$, which is impossible.
$*$ So $\|f(\overline{n}) - f(\overline{m})\| \leq \alpha_j - \eta$ or $\|g(\overline{n}) - g(\overline{m})\| \leq \beta_j - \eta$ for all $(\overline{n},\overline{m}) \in H_j(\mathbb{M}'_k)$.
Now we note that $H_j(\mathbb{M}'_k)$ can be identified with $[\mathbb{M}'_k]^{k+1}$ so, by Ramsey's Theorem, we get $\mathbb{M}' \in [\mathbb{M}'_k]^\omega$ such that $\|f(\overline{n}) - f(\overline{m})\| \leq \alpha_j - \eta$ for all $(\overline{n},\overline{m}) \in H_j(\mathbb{M}')$ or $\|g(\overline{n}) - g(\overline{m})\| \leq \beta_j - \eta$ for all $(\overline{n},\overline{m}) \in H_j(\mathbb{M}')$. This contradicts the definition of α_j or β_j.
Thus $\|\alpha_j e_1 + \beta_j e_2\| \leq \gamma_j$ for all $j \in \{1, \cdots, k\}$.
• By assumption, there exists $\mathbb{M}' \in [M'_k]^\omega$ so that

$$\|f(\overline{n}) - f(\overline{m})\| \leq \lambda \left(\sum_{j=1}^k (\alpha_j + \varepsilon')^p \right)^{\frac{1}{p}} \text{ and } \|g(\overline{n}) - g(\overline{m})\| \leq \lambda \left(\sum_{j=1}^k (\beta_j + \varepsilon')^p \right)^{\frac{1}{p}}$$

for all $(\overline{n},\overline{m}) \in I_k(\mathbb{M}')$. Let $x^n = (\alpha_n + \varepsilon')e_1 + (\beta_n + \varepsilon')e_2$ for each $n \in \{1, \cdots, k\}$. Using p-convexity, we get :

$$\begin{aligned}
\|h(\overline{n}) - h(\overline{m})\|^p &\leq \lambda^p \Big\| \left(\sum_{j=1}^k (\alpha_j + \varepsilon')^p \right)^{\frac{1}{p}} e_1 + \left(\sum_{j=1}^k (\beta_j + \varepsilon')^p \right)^{\frac{1}{p}} e_2 \Big\|^p \\
&\leq \lambda^p \sum_{n=1}^k \|x^n\|^p = \lambda^p \sum_{j=1}^k \|(\alpha_j + \varepsilon')e_1 + (\beta_j + \varepsilon')e_2\|^p \\
&\leq \lambda^p \sum_{j=1}^k (\gamma_j + 2\varepsilon')^p
\end{aligned}$$

for all $(\overline{n}, \overline{m}) \in I_k(\mathbb{M}')$.
Therefore,

$$\|h(\overline{n}) - h(\overline{m})\|^p \leq (\lambda + \varepsilon) \left(\sum_{j=1}^{k} \gamma_j^p \right)^{\frac{1}{p}},$$

for all $(\overline{n}, \overline{m}) \in I_k(\mathbb{M}')$, *i.e*, X has $(\lambda + \varepsilon)$-HIC$_{p,d}$. □

Remark 2.2.2. From this property about finite sums, we can deduce our main result. In order to do so, let us remark that a Banach space E that has a p-convex basis with constant 1 satisfies the following: if $x \in E$ and $(x_n)_{n \in \mathbb{N}}$ is a weakly null sequence in E, then

$$\limsup \|x + x_n\|^p \leq \|x\|^p + \limsup \|x_n\|^p.$$

Therefore, we deduce from the proof of Theorem 4.2 [67] that if E is in addition reflexive, then for every $k \in \mathbb{N}$, every $\mathbb{M} \in [\mathbb{N}]^\omega$, every $\varepsilon > 0$ and every Lipschitz function $f : ([\mathbb{M}]^k, d_{\mathbb{H}}) \to E$, there exist $\mathbb{M}' \in [\mathbb{M}]^\omega$ and $u \in E$ so that

$$\|f(\overline{n}) - u\| \leq \left(\sum_{j=1}^{k} \alpha_j^p \right)^{\frac{1}{p}} + \varepsilon$$

for all $\overline{n} \in [\mathbb{M}']^k$, where $\alpha_j = \sup_{(\overline{n}, \overline{m}) \in H_j(\mathbb{M})} \|f(\overline{n}) - f(\overline{m})\|$ for all $j \in \{1, \cdots, k\}$.

We now prove Theorem B stated in the introduction, that we recall here.

Theorem 2.2.3. *Let $p \in (1, \infty)$, $\lambda > 0$, $(X_n)_{n \in \mathbb{N}}$ a sequence of Banach spaces with property λ-HIC$_{p,d}$.*
Let E be a reflexive Banach space with a normalized 1-unconditional p-convex basis $(e_n)_{n \in \mathbb{N}}$ with convexity constant 1.
Then $X = \left(\sum_{n \in \mathbb{N}} X_n \right)_E$ has property $(\lambda + 2 + \varepsilon)$-HIC$_{p,d}$ for every $\varepsilon > 0$.

Proof. Let $\varepsilon > 0$, $\mathbb{M} \in [\mathbb{N}]^\omega$, $k \in \mathbb{N}$, $f : ([\mathbb{M}]^k, d_{\mathbb{H}}) \to X$ a Lipschitz function.
There exists $\varepsilon' > 0$ such that

$$(\lambda + 2 + \varepsilon') \left(\sum_{j=1}^{k} \alpha_j^p \right)^{\frac{1}{p}} + 4\varepsilon' \leq (\lambda + 2 + \varepsilon) \left(\sum_{j=1}^{k} \alpha_j^p \right)^{\frac{1}{p}}$$

where

$$\alpha_j = \sup_{(\overline{n}, \overline{m}) \in H_j(\mathbb{M})} \|f(\overline{n}) - f(\overline{m})\|$$

for all $j \in \{1, \cdots, k\}$.
The well-defined map

$$\phi : \begin{cases} X & \to \quad E \\ (x_n)_{n \in \mathbb{N}} & \mapsto \quad \sum_{n=1}^{\infty} \|x_n\| e_n \end{cases}$$

satisfies $\mathrm{Lip}(\phi) \leq 1$ and $\|\phi(x)\| = \|x\|$ for all $x \in X$, thus

$$\sup_{(\overline{n}, \overline{m}) \in H_j(\mathbb{M})} \|\phi \circ f(\overline{n}) - \phi \circ f(\overline{m})\| \leq \alpha_j$$

for every $j \in \{1, \cdots, k\}$.
From the previous remark, we get $u \in E$ and $\mathbb{M}' \in [\mathbb{M}]^\omega$ such that

$$\|\phi \circ f(\overline{n}) - u\| \leq \left(\sum_{j=1}^{k} \alpha_j^p\right)^{\frac{1}{p}} + \varepsilon'$$

for all $\overline{n} \in [\mathbb{M}']^k$. Let $N \in \mathbb{N}$ such that $\left\| \sum_{k=N+1}^{\infty} \|u_k\| e_k \right\| \leq \varepsilon'$.
For each $n \in \mathbb{N}$, let us denote by P_n the projection from X onto X_n and Π_n the projection from X onto $(\sum_{k=1}^{n} X_k)_E$.
We have

$$\begin{aligned}
\left\| \sum_{n=N+1}^{\infty} \|P_n \circ f(\overline{n})\| e_n \right\| &\leq \left\| \sum_{n=N+1}^{\infty} \|P_n \circ f(\overline{n})\| e_n \right\| - \left\| \sum_{n=N+1}^{\infty} \|u_n\| e_n \right\| + \varepsilon' \\
&\leq \left\| \sum_{n=N+1}^{\infty} (\|P_n \circ f(\overline{n})\| - \|u_n\|) e_n \right\| + \varepsilon' \\
&\leq \|\phi \circ f(\overline{n}) - u\| + \varepsilon' \\
&\leq \left(\sum_{j=1}^{k} \alpha_j^p\right)^{\frac{1}{p}} + 2\varepsilon'
\end{aligned}$$

for all $\overline{n} \in [\mathbb{M}']^k$.
Moreover, according to Proposition 2.2.1, we get an infinite subset $\mathbb{M}'' \in [\mathbb{M}']^\omega$ such that

$$\|\Pi_N \circ f(\overline{n}) - \Pi_N \circ f(\overline{m})\| \leq (\lambda + \varepsilon') \left(\sum_{j=1}^{k} \alpha_j^p\right)^{\frac{1}{p}}$$

for all $(\overline{n}, \overline{m}) \in I_k(\mathbb{M}'')$.
We deduce

$$\begin{aligned}
\|f(\overline{n}) - f(\overline{m})\| &\leq \|\Pi_N(f(\overline{n}) - f(\overline{m}))\| + \|(I - \Pi_N) \circ f(\overline{n})\| + \|(I - \Pi_N) \circ f(\overline{m})\| \\
&\leq (\lambda + \varepsilon') \left(\sum_{j=1}^{k} \alpha_j^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{k} \alpha_j^p\right)^{\frac{1}{p}} + 2\varepsilon' + \left(\sum_{j=1}^{k} \alpha_j^p\right)^{\frac{1}{p}} + 2\varepsilon' \\
&\leq (\lambda + 2 + \varepsilon) \left(\sum_{j=1}^{k} \alpha_j^p\right)^{\frac{1}{p}}
\end{aligned}$$

for all $(\overline{n}, \overline{m}) \in I_k(\mathbb{M}'')$. The result follows. □

Remark 2.2.4. With this result and Proposition 2.1.9, we immediately deduce the following: if each X_n, $n \in \mathbb{N}$, has property λ-HC$_{p,d}$, then $(\sum_{n\in\mathbb{N}} X_n)_E$ has property $(2\lambda+2+\varepsilon)$-HC$_{p,d}$ for every $\varepsilon > 0$.

Once again, we chose to state this theorem with property HIC$_{p,d}$, but the result stays true for property HFC$_{p,d}$, with a similar proof.

Remark 2.2.5. • Of course, the condition that all spaces have property $\mathrm{HIC}_{p,d}$ with the same constant is essential because

$$([\mathbb{N}]^{<\omega}, d_{\mathbb{H}}) \underset{L}{\hookrightarrow} X_\omega = \left(\sum_{n=1}^{\infty} \ell_1^n(\ell_2)\right)_{\ell_2}$$

even though $\ell_1^n(\ell_2)$ has property $\mathrm{HFC}_{2,d}$ (it is reflexive and 2-AUS) for every $n \in \mathbb{N}$.
To see that, let us note that, for every $k \in \mathbb{N}$, $([\mathbb{N}]^{\leq k}, d_{\mathbb{H}})$ isometrically embeds into $\ell_1^k(\ell_2)$. Then, the barycentric gluing technique by Baudier (see [6]), which consists in pasting embeddings of balls of growing radii via good convex combinations, gives us a bi-Lipschitz embedding from $([\mathbb{N}]^{<\omega}, d_{\mathbb{H}})$ into X_ω.
• In [18], Braga asked the following (Problem 3.7): if a Banach space X has the Banach-Saks property, *i.e*, every bounded sequence in X admits a subsequence whose Cesàro means converge in norm, does it follow that $([\mathbb{N}]^{<\omega}, d_\Delta)$ does not Lipschitz embed into X? The answer to this question is negative. Indeed, let $(p_n)_{n\in\mathbb{N}} \subset (1,\infty)$ be a decreasing sequence such that $\lim_{n\to+\infty} p_n = 1$ and let $X = (\sum_{n=1}^{\infty} \ell_{p_n})_{\ell_2}$. The space X has the Banach-Saks property (see [87]). With a similar argument, or simply appealing to Ribe's Theorem [96] (that states that X is uniformly homeomorphic to $X \oplus \ell_1$ and thus implies that ℓ_1 coarse Lipschitz embeds into X), we can see that

$$([\mathbb{N}]^{<\omega}, d_{\mathbb{H}}) \underset{L}{\hookrightarrow} X \text{ and } ([\mathbb{N}]^{<\omega}, d_\Delta) \underset{L}{\hookrightarrow} X.$$

Before we write a direct consequence of this theorem, let us briefly recall the definition of the James sequence spaces.
Let $p \in (1,\infty)$. The James space J_p is the real Banach space of all sequences $x = (x(n))_{n\in\mathbb{N}}$ of real numbers with finite p-variation and verifying $\lim_{n\to\infty} x(n) = 0$. The space J_p is endowed with the following norm

$$\|x\|_{\mathrm{J}_p} = \sup\left\{\left(\sum_{i=1}^{k-1} |x(p_{i+1}) - x(p_i)|^p\right)^{\frac{1}{p}} ; 1 \leq p_1 < p_2 < \cdots < p_k\right\}.$$

The space $\mathrm{J} = \mathrm{J}_2$, constructed by James in [55], is the historical example of a quasi-reflexive Banach space which is isomorphic to its bidual. In fact, J_p^{**} can be seen as the space of all sequences of real numbers with finite p-variation, which is $\mathrm{J}_p \oplus \mathbb{R}e$, where e denotes the constant sequence equal to 1.
Besides of being quasi-reflexive, the space J_p has the property of being p-AUS-able (see [84], Proposition 2.3) and its dual J_p^* is q-AUS-able, where q denotes the conjugate exponent of p (see [73] and references therein).
We can now state the following corollary.

Corollary 2.2.6. *Let $p, q \in (1,\infty)$, p' the conjugate exponent of p, $s = \min(p,q)$ and $t = \min(p',q)$.*
If X is a quasi-reflexive Banach space in A_p, then the space $\ell_q(X)$ has property $HC_{s,d}$.
In particular, $\ell_q(\mathrm{J}_p)$ has property $HC_{s,d}$ and $\ell_q(\mathrm{J}_p^*)$ has property $HC_{t,d}$.

Let us mention that we stated this corollary for ℓ_q-sums but we could have done it with any reflexive q-convexification of a Banach space with a 1-unconditional basis (such as

T_q, the q-convexification of Tsirelson space, or S_q, the q-convexification of Schlumprecht space, see [18] and references therein).

With $p = 2$, we get that the spaces $\ell_2(\mathrm{J})$ and $\ell_2(\mathrm{J}^*)$ have property HC_2 and thus cannot contain equi-Lipschitz copies of Hamming graphs. In fact, property HC_p provides more information than an obstruction to the equi-Lipschitz embedding of Hamming graphs, it also gives us an estimation of some compression exponents, given by the result below. Before stating it, we need the following definitions.

Definition 2.2.7. Let X be a Banach space. Following Milman (see [79]), we introduce the following modulus: for all $t \geq 0$, let

$$\overline{\delta}_X(t) = \inf_{x \in S_X} \sup_{Y} \inf_{y \in S_Y} (\|x + ty\| - 1)$$

where Y runs through all closed linear subspaces of X of finite codimension.
We say that $\|\cdot\|$ is *asymptotically uniformly convex* (in short AUC) if $\overline{\delta}_X(t) > 0$ for all $t > 0$. If $q \in [1, \infty)$, $\|\cdot\|$ is said to be *q-AUC* if there is a constant $C > 0$ such that, for all $t \in [0, 1]$, $\overline{\delta}_X(t) \geq Ct^q$.

Definition 2.2.8. Let $q \in (1, \infty)$ and X be a Banach space. We say that X has the *q-co-Banach-Saks property* if for every semi-normalized weakly null sequence $(x_n)_{n\in\mathbb{N}}$ in X, there exist a subsequence $(x'_n)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ and $c > 0$ such that, for all $k \in \mathbb{N}$ and all $k \leq n_1 < \cdots < n_k$, we have

$$\|x'_{n_1} + \cdots + x'_{n_k}\| \geq ck^{1/q}.$$

Theorem 2.2.9. *Let $q < p$ in $(1, \infty)$. Assume X is an infinite-dimensional Banach space with the q-co-Banach-Saks property and Y is a Banach space with property HC_p. Then X does not coarse Lipschitz embed into Y. More precisely, the compression exponent $\alpha_Y(X)$ of X into Y satisfies the following:*
(i) if X contains an isomorphic copy of ℓ_1, then $\alpha_Y(X) \leq \frac{1}{p}$;
(ii) otherwise, $\alpha_Y(X) \leq \frac{q}{p}$.
In particular, if X is q-AUC, then $\alpha_Y(X) \leq \frac{q}{p}$.

We refer the reader to Theorem 3.5 and Corollary 3.6 of [74] for a proof of this result.
Let us note that Proposition 3.2 of [74] also stays true by replacing "quasi-reflexive AUS" by "having property HC_p for some $p \in (1, \infty)$".

We also would like to mention the following: we could define symmetric concentration properties SFC_p, SIP_p and SC_p, corresponding respectively to properties HFC_p, HIC_p and HC_p by asking the function f to be Lipschitz for the symmetric distance in the definitions of these properties (instead of being Lipschitz for the Hamming distance). Then, it is known that a reflexive (resp. quasi-reflexive) p-AUS Banach space, for $p \in (1, \infty)$, would have property SFC_p (resp. SC_p). Moreover, even though we wrote our properties $\mathrm{HFC}_{p,d}$, $\mathrm{HIC}_{p,d}$ and $\mathrm{HC}_{p,d}$ with the letter "H" because the quantities α_j, $j \in \{1, \cdots, k\}$ can be seen as directional Lipschitz constants when $[\mathbb{N}]^k$ is endowed with the Hamming distance, we could replace "$f : ([\mathbb{N}]^k, d_{\mathbb{H}}) \to X$ Lipschitz" by "$f : [\mathbb{N}]^k \to X$ bounded" in the definitions so that no reference to any specific metric is made. With that remark in mind and the fact that $HC_{p,d}$ implies SC_p, we get that property $HC_{p,d}$ prevents the equi-Lipschitz embeddings of the symmetric graphs.

Before concluding this subsection with a last result, let us recall some facts that we will use concerning the spaces Ti_q^*, the dual of the q-convexification of the Tirilman space Ti (see [99], [23], [28] and references therein for more information about this space), and S_q^*, the dual of the q-convexification of Schlumprecht space S (see [97], [28] and references therein for more information about this space).
If we denote $(e_n^*)_{n\in\mathbb{N}}$ the coordinate functionals associated with the canonical basis $(e_n)_{n\in\mathbb{N}}$ of Ti, it is known that $(e_n^*)_{n\in\mathbb{N}}$ is 1-symmetric and that the formal identity $I : \ell_q \to \mathrm{Ti}_q$ is bounded and stricty singular (see [28]). As for S_q^*, if we denote $(f_n^*)_{n\in\mathbb{N}}$ the sequence of coordinate functionals associated with the canonical basis $(f_n)_{n\in\mathbb{N}}$ of S and if p and q are conjugate exponents, it is known (see [28, Proposition 6.5 (iv)]) that for any finite non-empty subset E of $\mathbb{N}$,

$$\Big\| \sum_{i\in E} f_i^* \Big\|_{S_q^*} \geq |E|^{1/p} \log_2(|E|+1)^{1/q},$$

and that $(f_n^*)_{n\in\mathbb{N}}$ is 1-subsymmetric.

Proposition 2.2.10. *Let $p \in (1,\infty)$, q its conjugate exponent.*
The space Ti_q^* has property HFC_p but does not have property $HC_{p,d}$, and the space S_q^* has property $HFC_{s,d}$ for every $s \in (1,p)$, but does not have property HC_p.

Proof. This space Ti_q^* is reflexive and $\mathrm{Ti}_q^* \in \mathrm{N}_p$ (see [28] Proposition 6.5 (v)) hence it has property HFC_p. Now, for $a = (a_j)_{j=1}^k \in B_{\ell_p^k}$, we let $f : \begin{cases} [\mathbb{N}]^k & \to & \mathrm{Ti}_q^* \\ \overline{n} & \mapsto & \sum_{j=1}^k a_j e_{n_j}^* \end{cases}$. This map satisfies $\mathrm{Lip}(f) \leq 2$.
Let us assume Ti_q^* has property λ-$\mathrm{HC}_{p,d}$ for some $\lambda > 0$. Then, there exist $\overline{n}, \overline{m} \in [\mathbb{N}]^k$ such that $\overline{n} \cap \overline{m} = \varnothing$ and

$$\Big\| \sum_{j=1}^k a_j e_j^* \Big\| = \|f(\overline{n})\| \leq \|f(\overline{n}) - f(\overline{m})\| \leq 2\lambda$$

because of the 1-symmetry of $(e_j^*)_{j\in\mathbb{N}}$. We deduce that the sequence of coordinate functionals $(e_j^*)_{j\in\mathbb{N}}$ is dominated by the ℓ_p basis. This is impossible, by the same argument used by Causey [28] to prove $\mathrm{Ti}_q^* \notin \mathrm{A}_p$ (by duality, the Ti_q basis would dominate, and would therefore be equivalent to the ℓ_q basis, contradicting the strict singularity of the formal inclusion $I : \ell_q \to \mathrm{Ti}_q$, cf. [28, Proposition 6.5 (ii)]).
Then, the space S_q^* is reflexive and $S_q^* \in \mathrm{P}_p$ (see the proof of Theorems 6.2, 6.3, case $\xi = 0$, and Remark 6.7 in [28]) hence it has property $\mathrm{HFC}_{s,d}$ for every $s \in (1,p)$. We define similarly, for $a = (a_j)_{j=1}^k \in B_{\ell_p^k}$, a 2-Lipschitz map $f : \begin{cases} [\mathbb{N}]^k & \to & S_q^* \\ \overline{n} & \mapsto & \sum_{j=1}^k a_j f_{n_j}^* \end{cases}$.
Now, we may argue as we did for Ti_q^* to deduce that, for all $(\overline{n},\overline{m}) \in I_k(\mathbb{N})$:

$$\|f(\overline{n}) - f(\overline{m})\| \geq k^{1/p} \log_2(k+1)^{1/q}$$

because of the 1-subsymmetry of the canonical basis and [28, Proposition 6.5 (iv)]. Since $\lim_{k\to\infty} \log_2(k+1)^{1/q} = \infty$, the space S_q^* cannot have property HC_p. □

2.2.2 Related questions

The following questions about Theorem B come up naturally.

Problem 2.2.11. Can we replace property $\mathrm{HFC}_{p,d}$ (resp. $\mathrm{HIC}_{p,d}$) by property HFC_p (resp. HIC_p) in Theorem B?

Problem 2.2.12. Can the conclusion of Theorem B be improved so that $X = \left(\sum_{n\in\mathbb{N}} X_n\right)_E$ has property $(\lambda+\varepsilon)$-$\mathrm{HIC}_{p,d}$ for every $\varepsilon > 0$?

Problem 2.2.13. If a Banach space X has property $\mathrm{HFC}_{p,d}$ (or $\mathrm{HIC}_{p,d}$), does $L_p(X)$ have some concentration property?

Problem 2.2.14. Let $p \in (1,\infty)$, X a p-AUS Banach space so that X is complemented in X^{**} and that X^{**}/X is reflexive and p-AUS-able. Does X have property HC_p?

A positive answer to the second question would provide us, for each $p \in (1,\infty)$, with an example of a reflexive Banach space, not AUS-able, with property $\mathrm{HFC}_{p,d}$. Indeed, following Braga's proof of Theorem 7.1 [19], the space $X_{p,\ell_1,T}$ would be such an example (see [19] and references therein for more information about this space).

Moreover, let us recall that Kalton proved the existence of a Banach space X that is not p-AUS-able but that is uniformly homeomorphic to a p-AUS Banach space (see [64]). Thus, the space X has property $\mathrm{HFC}_{p,d}$, $p \in (1,\infty)$, even though it is not p-AUS-able. However, the following problem remains open.

Problem 2.2.15. Is there a Banach space that has property HFC_p (or $\mathrm{HFC}_{p,d}/\mathrm{HC}_p/\mathrm{HC}_{p,d}$) without being AUS-able? If a Banach space X coarse Lipschitz embeds into a Banach space Y that is reflexive and AUS, does it follow that X is AUS-able?

We will finish this section by saying a few words about a natural class of spaces to study here: the Lindenstrauss spaces (see [75]). For any separable Banach space X, we will denote by Z_X the Lindenstrauss space associated to X, constructed so that the quotient Z_X^{**}/Z_X is linearly isomorphic to X. In [19], Braga showed that neither $Z_{c_0}^*$, Z_{ℓ_1} or $Z_{X_\omega^*}^*$ can have any of the concentration properties we introduced, even though they are 2-AUS-able (see [31]) and do not contain c_0 nor ℓ_1. The key point of the proof for the spaces $Z_{c_0}^*$ and $Z_{X_\omega^*}^*$ is that they satisfy the assumptions of the following proposition, that can be deduced from [19].

Proposition 2.2.16. *Let X be a Banach space such that X^* is separable.*
Assume that there exist $A, C \geq 1$, $(z_{k,j,n}^{**})_{k\in\mathbb{N}, j\in\{1,\cdots,k\}, n\in\mathbb{N}} \subset CB_{X^{**}}$ such that for every $k \in \mathbb{N}$, the map

$$F_k : \begin{cases} [\mathbb{N}]^k & \to & X^{**} \\ \overline{n} & \mapsto & \sum_{j=1}^{k} z_{k,j,n_j}^{**} \end{cases}$$

satisfies

$$\frac{1}{A} d_{\mathbb{H}}(\overline{n},\overline{m}) \leq \|F_k(\overline{n}) - F_k(\overline{m})\| \leq A d_{\mathbb{H}}(\overline{n},\overline{m})$$

for all $\overline{n},\overline{m} \in [\mathbb{N}]^k$.
Then, the space X does not have any of the concentration properties introduced before.

We can therefore ask ourselves the following question.

Problem 2.2.17. Can we find an infinite-dimensional Banach space X and a $p \in (1, \infty)$ such that Z_X or Z_X^* has property HC_p?

Finally, by Aharoni's Theorem [1], we know that the Hamming graphs equi-Lipschitz embed into $Z_{c_0}^{**}/Z_{c_0}$. Does it mean that these graphs can be Lipschitz embedded into $Z_{c_0}^{**}$? Into Z_{c_0}?

2.3 Asymptotic-c_0 spaces

Before stating the last result of this chapter, we recall the definition of an asymptotic-c_0 space, *i.e* a space that is in the class A_∞ introduced in Chapter 1. The following definition is due to Maurey, Milman and Tomczak-Jaegermann [78].

Definition 2.3.1. Let X be a Banach space. We denote by $\mathrm{cof}(X)$ the set of all its closed finite-codimensional subspaces.
For $C \geq 1$, we say that X is C-*asymptotically* c_0 if, for any $k \in \mathbb{N}$, we have

$$\exists X_1 \in \mathrm{cof}(X)\, \forall x_1 \in S_{X_1}\, \exists X_2 \in \mathrm{cof}(X)\, \forall x_2 \in S_{X_2}\, \cdots\, \exists X_k \in \mathrm{cof}(X)\, \forall x_k \in S_{X_k},$$

$$\forall (a_1, \dots, a_k) \in \mathbb{R}^k, \Big\| \sum_{i=1}^{k} a_i x_i \Big\| \leq C \max_{1 \leq i \leq k} |a_i|$$

We say that X is *asymptotically* c_0 (or asymptotic-c_0) if it is C-asymptotically c_0 for some $C \geq 1$.

Let X be a Banach space. A family $(x_j^{(i)}; i, j \in \mathbb{N}) \subset X$ is called an *infinite array*. For an infinite array $(x_j^{(i)}; i, j \in \mathbb{N})$, we call the sequence $(x_j^{(i)})_{j \in \mathbb{N}}$ the i-*th row of the array*. We call an array *weakly null* if all rows are weakly null. A *subarray* of $(x_j^{(i)}; i, j \in \mathbb{N})$ is an infinite array of the form $(x_{j_s}^{(i)}; i, s \in \mathbb{N})$, where $(j_s) \subset \mathbb{N}$ is a subsequence. Thus, for a subarray, we are taking the same subsequence in each row.

The following notion, introduced by Halbeisen and Odell ([51]), is a generalization of spreading models.

Definition 2.3.2. A basic sequence $(e_i)_{i \in \mathbb{N}}$ is called an *asymptotic model* of a Banach space X if there exist an infinite array $(x_j^{(i)}; i, j \in \mathbb{N}) \subset S_X$ and a null-sequence $(\varepsilon_n)_{n \in \mathbb{N}} \subset (0, 1)$, so that, for all $n \in \mathbb{N}$, all $(a_i)_{i=1}^n \subset [-1, 1]$ and all $n \leq k_1 < k_2 < \cdots < k_n$,

$$\Big| \| \sum_{i=1}^{n} a_i x_{k_i}^{(i)} \| - \| \sum_{i=1}^{n} a_i e_i \| \Big| < \varepsilon_n.$$

The following proposition concerning this notion was proved in [51].

Proposition 2.3.3 ([51], Proposition 4.1 and Remark 4.7.5)**.** *Assume that $(x_j^{(i)}; i, j \in \mathbb{N}) \subset S_X$ is an infinite array, all of whose rows are normalized and weakly null. Then there is a subarray of $(x_j^{(i)}; i, j \in \mathbb{N})$ which has a 1-suppression-unconditional asymptotic model $(e_i)_{i \in \mathbb{N}}$.*

We call a basic sequence $(e_i)_{i\in\mathbb{N}}$ *c-suppression-unconditional*, for some $c \geq 1$, if, for any $(a_i)_{i\in\mathbb{N}} \subset c_{00}$ and any $A \subset \mathbb{N}$, we have :

$$\Big\| \sum_{i\in A} a_i e_i \Big\| \leq c \Big\| \sum_{i=1}^{\infty} a_i e_i \Big\|.$$

Note that a c-unconditional basic sequence is c-suppression-unconditional and a c-suppression-unconditional basic sequence is $2c$-unconditional.

As for the proof of the fact that every Banach space with property HFC_∞ is asymptotic-c_0 (see [11]), the key ingredient will be the following theorem of Freeman, Odell, Sari and Zheng.

Theorem 2.3.4 ([41], Theorem 4.6)**.** *If a separable Banach space X does not contain any isomorphic copy of ℓ_1 and all the asymptotic models generated by normalized weakly null arrays are equivalent to the c_0 unit vector basis, then X is asymptotically c_0.*

We now have all the tools to prove our result.

Theorem 2.3.5. *If a Banach space has property HC_∞, then it is asymptotic-c_0.*

Proof. Let X be a Banach space with property HC_∞. Then X has property λ-HIC_∞, for some $\lambda > 0$, by Proposition 2.1.9. Let us note that we can assume that X is separable by Proposition 11 of [36], and that X cannot contain an isomorphic copy of ℓ_1 since ℓ_1 does not have this property.
Assume by contradiction that X is not asymptotic-c_0. By Theorem 2.3.4, there exists a 1-suppression-unconditional sequence $(e_i)_{i\in\mathbb{N}}$ that is not equivalent to the unit vector basis of c_0, and hence $\lambda_k = \Big\| \sum_{i=1}^k (-1)^i e_i \Big\| \nearrow \infty$, if $k \nearrow \infty$, and that is generated as an asymptotic model of a normalized weakly null array $(x_j^{(i)}; i, j \in \mathbb{N})$ in X. Let $k \in \mathbb{N}$ such that $\dfrac{\lambda_{2k}}{4} > \lambda$, and $\delta = \dfrac{\lambda_{2k}}{2}$. After passing to appropriate subsequences of the array, we may assume that, for any $2k \leq j_1 < \cdots < j_{2k}$ and any $a_1, \ldots, a_{2k} \in [-1, 1]$, we have

$$\left| \Big\| \sum_{i=1}^{2k} a_i x_{j_i}^{(i)} \Big\| - \Big\| \sum_{i=1}^{2k} a_i e_i \Big\| \right| < \delta. \tag{2.1}$$

Define now $f(\overline{m}) = \frac{1}{2} \sum_{i=1}^k x_{m_i}^{(i)}$ for $\overline{m} = (m_1, \ldots, m_k) \in [\mathbb{N}]^k$. Note that f is 1-Lipschitz for the metric $d_{\mathbb{H}}$.
Let $\mathbb{M} \in [\mathbb{N}]^\omega$, $\overline{n}, \overline{m} \in [\mathbb{M}]^k$ such that $((n_1, \ldots, n_k), (m_1, \ldots, m_k)) \in I_k(\mathbb{M})$ and $m_1, n_1 > 2k$. Using equation (2.1), we get that :

$$\|f(m_1, \ldots, m_k) - f(n_1, \ldots, n_k)\| \geq \frac{1}{2}\lambda_{2k} - \frac{\delta}{2} = \frac{\lambda_{2k}}{4} > \lambda.$$

This contradicts the assumption on X. The result follows. □

Theorem B and Corollary 2.2.6 provided us with examples of non-quasi-reflexive Banach spaces having property HC_p, for $p \in (1, \infty)$. In order to obtain a similar result with $p = \infty$, it seems natural to consider a T^*-sum of spaces with λ-HFC_∞, $\lambda > 0$. However, as a direct consequence of Theorem 2.3.5, we get the following corollary.

Corollary 2.3.6. *The space $T^*(T^*)$, where T^* is the original Banach space constructed by Tsirelson in [98], does not have property HC_∞ although it has property $HFC_{p,d}$ for every $p \in (1, \infty)$.*

Before proving this corollary, let us recall some properties of the space T^*. First, it is reflexive, so we will denote its dual by T. This space T^* has a normalized, shrinking, 1-unconditional basis $(e_n)_{n=1}^\infty$. Let us denote by $(e_n^*)_{n\in\mathbb{N}} \subset T$ the coordinate functionals of $(e_n)_{n\in\mathbb{N}}$. For an element $x = \sum_{n=1}^\infty e_n^*(x)e_n \in X$, let us denote by $\mathrm{Supp}(x)$ the support of x, *i.e*, the subset of integers n such that $e_n^*(x) \neq 0$. The space T satisfies the following (see [98] and [39]): for every $(x_i)_{i=1}^n \subset T$ with $(\mathrm{Supp}(x_i))_{i=1}^n$ increasing (*i.e* the supports are disjoint and consecutive), $\|x_i\| = 1$, and $\mathrm{Supp}(x_1) \subset [k+1, \infty)$, we have

$$\forall (a_i)_{i=1}^n \subset \mathbb{R}, \Big\| \sum_{i=1}^n a_i x_i \Big\| \geq \frac{1}{2} \sum_{i=1}^n |a_i|.$$

Proof of Corollary 2.3.6. The lack of property HC_∞ is a direct consequence of Lemma 2.7 of [9] (asserting that $T^*(T^*)$ is not asymptotic-c_0) and Theorem 2.3.5 above. The fact that $T^*(T^*)$ has property $\mathrm{HFC}_{p,d}$, for every $p \in (1, \infty)$, can be deduced from Theorem 5.9 of [35], applied with $p = 1$, $X_n = T^*$ for all $n \in \mathbb{N}$ and $\mathfrak{E} = T^*$. Indeed, if we let

$$\mathrm{p}(X) = \inf \left\{ q \geq 1;\ X \text{ is } p\text{-AUS-able}, \frac{1}{p} + \frac{1}{q} = 1 \right\}$$

for a given Banach space X, Theorem 5.9 [35] asserts that $\mathrm{p}(T^*(T^*)) = 1$. As $T^*(T^*)$ is reflexive, the result follows from Theorem 6.1 [67] of Kalton and Randrianarivony (see the remark after Definition 2.5). □

As property HC_∞ prevents the equi-coarse embeddability of the Hamming graphs, we can therefore ask the following :

Problem 2.3.7. If the Hamming graphs do not equi-coarsely embed into a Banach space X, does it follow that X is asymptotic-c_0?

Moreover, let us note that we only defined property $\mathrm{HFC}_{p,d}$ (resp. $\mathrm{HIC}_{p,d}$, resp. $\mathrm{HC}_{p,d}$) for $p \in (1, \infty)$ because, for $p = \infty$, this is exactly the definition of property HFC_∞ (resp. HIC_∞, resp. HC_∞). In the light of Section 2.2, the following question seems natural.

Problem 2.3.8. Does property HC_∞ imply quasi-reflexivity?

In addition, as Lindenstrauss spaces provided us with non trivial examples of AUS-able dual Banach spaces without any concentration property, the following result, due to Schlumprecht, provides us with a non trivial example of an asymptotic-c_0 separable dual Banach space without any concentration property.

Theorem 2.3.9 (Schlumprecht)**.** *Let X be a Banach space whose dual is separable. Then, there exists an asymptotic-c_0 separable dual Banach space Z_X such that*

$$Z_X^{**} = Z_X \oplus X^*.$$

With this theorem, proved in Section 5 of [40], and arguments of Braga [19], we can prove the following result, which proof is in the spirit of Proposition 2.2.16.

Corollary 2.3.10. *There exists a separable asymptotic-c_0 dual space Z such that there exists a sequence of equi-Lipschitz functions $(f_k : [\mathbb{N}]^{2k} \to Z)_{k\in\mathbb{N}}$ satisfying the following property: for all $\varepsilon > 0$, all $k \in \mathbb{N}$ and all $\overline{n}, \overline{m} \in [\mathbb{N}]^k$, there exists $i \geq \max(n_k, m_k)$ such that*

$$\|f_k(\overline{n}, \overline{n}') - f_k(\overline{m}, \overline{m}')\| \geq 2d_\Delta(\overline{n}, \overline{m}) - \varepsilon$$

for all $\overline{n}', \overline{m}' \in [\mathbb{N}]^k$ with $n_1', m_1' > i$.
In particular, Z cannot have any of the concentration properties we introduced.

Proof. Let $Z = Z_{c_0}$ given by the previous theorem. It is a separable asymptotic-c_0 dual space. Now, we start by noting that ℓ_1 linearly embeds into Z^{**} hence the existence of a bounded sequence $(z_n^{**})_{n\in\mathbb{N}} \subset Z^{**}$ with the following property

$$(*) \quad \forall k \in \mathbb{N}, \forall(\varepsilon_1, \ldots, \varepsilon_k) \in \{\pm 1\}^k, \forall(n_1, \ldots, n_k) \in [\mathbb{N}]^k, \Big\|\sum_{j=1}^k \varepsilon_j z_{n_j}^{**}\Big\| \geq k.$$

Let $C = \sup_{n\in\mathbb{N}} \|z_n^{**}\|_{Z^{**}}$. Since Z^* is separable, by Goldstine's Theorem, for each $n \in \mathbb{N}$, we can find a sequence $(z_{(n,m)})_{m\in\mathbb{N}} \subset CB_Z$ such that

$$z_n^{**} = \omega^* - \lim_{m\to\infty} z_{(n,m)}.$$

Then, for each $k \in \mathbb{N}$, the map

$$f_k : \begin{cases} [\mathbb{N}]^{2k} & \to \quad Z \\ \overline{n} & \mapsto \quad \sum_{j=1}^k z_{(n_j, n_{k+j})} \end{cases}$$

satisfies $\mathrm{Lip}(f_k) \leq 2C$. Using weak*-lower semicontinuity of the norm and $(*)$, we get the result.
In particular, this sequence of equi-Lipschitz functions is such that

$$\forall k \in \mathbb{N}, \forall \mathbb{M} \in [\mathbb{N}]^\omega, \exists(\overline{n}, \overline{m}) \in I_{2k}(\mathbb{M}); \|f_k(\overline{n}) - f_k(\overline{m})\| \geq 2k - 1$$

thus Z cannot have any of the concentration properties we introduced. □

Remark 2.3.11. Let us mention that Z is a non-quasi-reflexive asymptotic-c_0 space that does not satisfy any of the concentration properties for non-trivial reasons. Indeed, since Z has the Radon-Nikodỳm property (see [37]) and a separable bidual, it cannot contain a linear copy of c_0, not even a Lipschitz copy of ℓ_1.

Chapter 3

Three-space properties and non-linear stabilities

This is part of a joint work with Ryan Causey and Gilles Lancien (see [30]). This chapter deals with two different types of results. The first kind concerns the three-space problem and the second kind is about non-linear stability.

In a first section, we state and prove those dealing with the three-space problem. We start by reducing the questions to the separable setting and giving a shorter proof of a past result by Causey, Draga and Kochanek. Next, with a single counterexample, we prove that neither T_p, A_p or N_p, $1 < p < \infty$, is a three-space property and we prove that the assumption of separability can be removed from the result of Johnson and Zippin that "being separable and in T_∞ is a three-space property". We finish this section with the proof of our main result, the fact that being asymptotic-c_0, *i.e* being in A_∞, is also one. That finishes to solve the three-space problem for the family of asymptotic properties considered in Chapters 1 and 3.

In a second section, after reminding the reader the previous known results about non-linear stability concerning the properties at stake in those chapters, we use the characterization of A_p we proved in the first chapter to deduce the stability of this class under coarse-Lipschitz equivalences, when $p \in (1, \infty)$. We finish this chapter by gathering a few examples of T_∞ or A_∞ spaces and related problems.

3.1 Three-space properties

3.1.1 Introduction

We recall that a property (P) of Banach spaces is a *Three-Space Property* (3SP in short) if it passes to quotients and subspaces and a Banach space X has (P) whenever it admits a subspace Y such that Y and X/Y have (P).

Note first that the linear properties considered in this chapter pass easily to subspaces and quotients.

Proposition 3.1.1. *Fix $1 < p \leqslant \infty$ and $X \in \mathsf{Ban}$. If X is in any of the classes $\mathsf{T}_p, \mathsf{A}_p, \mathsf{N}_p$, or P_p, then any subspace, quotient, or isomorph of X lies in the same class.*

Proof. For subspaces and isomorphs, the result is clear. For quotients, the result follows easily from the dual characterizations of these properties, which clearly pass to weak*-closed subspaces of X^* (we recall that $(X/Y)^*$ is canonically isometric to $Y^\perp \subset X^*$ by a weak*-weak*-bi-continuous map). □

The following lemma will allow us, when convenient, to reduce our questions to the separable setting.

Lemma 3.1.2. *Let* I *be a class of Banach spaces which contains all subspaces, quotients, and isomorphs of its members. Suppose also that membership in* $\mathsf{I} \cap \mathsf{Sep}$ *is a* 3*SP, and that if* X *is a Banach space such that every separable subspace of* X *lies in* I*, then* X *lies in* I*. Then membership in* I *is a* 3*SP.*

Proof. Let X be a Banach space and suppose that Y is a subspace of X such that $Y, X/Y \in \mathsf{I}$. If X is not in I, then there exists a separable subspace E of X such that E is not in I. Fix a countable, dense subset S of E and for each $x \in S$, fix a countable subset R_x of Y such that

$$\|x\|_{X/Y} = \inf_{y \in R_x} \|x - y\|.$$

Let G denote the closed linear span of

$$E \cup \bigcup_{x \in S} R_x$$

and let F denote the closed linear span of $\bigcup_{x\in S} R_x$. Then F, G are separable and F, being a subspace of Y, lies in $\mathsf{I} \cap \mathsf{Sep}$. Moreover, it follows from the construction of G that G/F is isometric to a subspace of X/Y, which means G/F also lies in $\mathsf{I} \cap \mathsf{Sep}$. Therefore G lies in I, as does $E \leqslant G$. Therefore every separable subspace of X lies in I, as does X. □

3.1.2 Past results

It was shown by Causey, Draga, Kochanek in [29] that membership in P_p is a 3SP, although it was not stated in this way. We isolate here a shorter and more direct argument. We will show the following.

Theorem 3.1.3. *Fix a Banach space* X*, a closed subspace* Y *of* X*, and* $1 < p \leqslant \infty$*.*

(i) If $\mathsf{n}_p(Y)$ *and* $\mathsf{n}_p(X/Y)$ *are finite, then there exist constants* C, λ *such that for all* $2 \leqslant n \in \mathbb{N}$*,* $\mathsf{n}_p(X) \leqslant C(\log n)^\lambda$*.*

(ii) If Y *and* X/Y *have* P_p*, so does* X*.*

The fact that P_p is a 3SP was shown in Theorem 7.5 of [29]. The proof there established an inequality similar to Theorem 3.1.3(i), but using a_p rather than n_p. In fact, the argument there was given for asymptotic Rademacher type p, which deals with Rademacher averages of arbitrary linear combinations of the branches of weakly null trees, which added significant technicality to the proof. Because n_p deals only with flat linear combinations, we sketch the simpler proof below.

We will use the following, which is an analogue of a lemma of Enflo, Lindenstrauss, and Pisier in their solution of the Palais problem [38].

Lemma 3.1.4. *For any Banach space X, any closed subspace Y of X, and any $m, n \in \mathbb{N}$,*

$$\theta_{mn}(X) \leqslant 6\big(\theta_m(X/Y)\theta_n(X) + \theta_m(X)\theta_n(Y)\big)$$

(we recall that $\theta_n(X)$ was defined before Remark 1.2.3).

Let us first deduce Theorem 3.1.3 from Lemma 3.1.4.

Proof of Theorem 3.1.3. (i) By Lemma 3.1.4, for any $n \in \mathbb{N}$,

$$\begin{aligned}\mathsf{n}_{p,n^2}(X) = \frac{\theta_{n^2}(X)}{n^{2/p}} &\leqslant 6\Big(\frac{\theta_n(X/Y)}{n^{1/p}} + \frac{\theta_n(Y)}{n^{1/p}}\Big)\frac{\theta_n(X)}{n^{1/p}} \\ &\leqslant c\mathsf{n}_{p,n}(X),\end{aligned}$$

where $c = 6(\mathsf{n}_p(X/Y) + \mathsf{n}_p(Y))$. We argue as in Theorem 3 of [38] to deduce the existence of the constants C and λ.

(ii) Assume $Y, X/Y$ have P_p. Fix $1 < r < s < p$. Since $Y, X/Y$ have P_p, they also have N_s, which means there exist constants C, λ such that for all $2 \leqslant n$, $\theta_n(X) = \mathsf{n}_{s,n}(X)n^{1/s} \leqslant C(\log n)^\lambda n^{1/s}$. Then $\mathsf{n}_{r,n}(X) = \theta_n(X)n^{-1/r} \leqslant C(\log n)^\lambda n^{1/s-1/r}$, which vanishes as n tends to infinity. Therefore, for $1 < r < p$, $\mathsf{n}_r(X) < \infty$, so $X \in \mathsf{P}_p = \bigcap_{1<r<p} \mathsf{N}_r$ (by Theorem 1.2.13).

□

We next recall an easy technical piece which we will need for the proof of Lemma 3.1.4.

Claim 1. *Let X be a Banach space and Y a closed subspace. For any weak neighborhood U_1 of 0 in X and $R, \delta > 0$, there exists a weak neighborhood U_2 of 0 in X such that if $x \in U_2 \cap RB_X$ with $\|x\|_{X/Y} < \delta$, then there exists $y \in U_1 \cap RB_Y$ such that $\|x - y\| < 6\delta$.*

Proof. If it were not so, then for some weak neighborhood U_1 of 0 in X and some $R, \delta > 0$, there would exist a weakly null net $(x_\lambda) \subset RB_X$ such that, for all λ, $\|x_\lambda\|_{X/Y} < \delta$ and for all $y \in U_1 \cap RB_Y$, $\|x_\lambda - y\| \geqslant 6\delta$. For each λ, we can fix $y_\lambda \in Y$ such that $\|x_\lambda - y_\lambda\| < \delta$. By passing to a subnet and relabeling, we can assume (y_λ) is weak*-convergent to some $y^{**} \in B_{X^{**}}$. Fix $\varepsilon > 0$ and a finite subset F of X^* such that $V := \{x \in X : (\forall x^* \in F)(|x^*(x)| < 2\varepsilon)\} \subset U_1$. Since (x_λ) is weakly null and (y_λ) is weak*-convergent to y^{**}, we can find λ_1, a finite subset G of the index set of (x_λ), and positive numbers $(w_\lambda)_{\lambda\in G}$ summing to 1 such that

(i) for all $x^* \in F$ and $\lambda \in \{\lambda_1\} \cup G$, $|y^{**}(x^*) - x^*(y_\lambda)| < \varepsilon$,

(ii) $\|\sum_{\lambda\in G} w_\lambda x_\lambda\| < \delta$.

Let $y_1 = y_{\lambda_1} - \sum_{\lambda\in G} w_\lambda y_\lambda \in V$ and note that

$$\|y_1 - x_{\lambda_1}\| \leqslant \|y_{\lambda_1} - x_{\lambda_1}\| + \sum_{\lambda\in G} w_\lambda\|y_\lambda - x_\lambda\| + \|\sum_{\lambda\in G} w_\lambda x_\lambda\| < 3\delta.$$

Since $\|x_{\lambda_1}\| \leqslant R$, $\|y_1\| \leqslant R + 3\delta$. If $\|y_1\| \leqslant R$, let $y = y_1$, and otherwise let $y = \frac{R}{\|y_1\|}y_1$, noting that $\|y - x_{\lambda_1}\| \leqslant \|y - y_1\| + \|y_1 - x_{\lambda_1}\| < 6\delta$. By convexity of V, $y \in V \subset U_1$, and we reach a contradiction.

□

Let us now sketch the proof of Lemma 3.1.4.

Sketch. If Y is finite dimensional, then $\theta_n(Y) = 0$ and $\theta_n(X/Y) = \theta_n(X)$ for all $n \in \mathbb{N}$. Then the inequality follows, without the factor of 6, using submultiplicativity of $\theta_n(X)$. A similar conclusion holds if X/Y is finite dimensional. We can therefore assume $Y, X/Y$ are infinite dimensional, and $\theta_n(Y), \theta_n(X/Y) \geq 1$ for all $n \in \mathbb{N}$.

The idea is to consider a weakly null tree indexed by D^{mn} that consists of inner trees of height m, and outer trees of height n (we do not define outer and inner trees but we hope the image will be clear to the reader after reading the construction below). For $\psi > \theta_m(X/Y)$, $\psi_1 > \theta_m(X)$, $\phi > \theta_n(Y)$, and $\phi_1 > \theta_n(X)$, we can fix winning strategies χ, χ_1, ϖ, and ϖ_1 for Player I in each of the games $\Theta(\psi, m)$ on X/Y, $\Theta(\psi_1, m)$ on X, $\Theta(\phi, n)$ on Y, and $\Theta(\phi_1, n)$ on X, respectively. For a weakly null collection $(x_t)_{t \in D^{\leqslant mn}} \subset B_X$, we claim that we can recursively select $t_1 \in D^m$, $y_1 \in Y$, $t_2 \in D^{2m}$ such that $t_1 \prec t_2$, $y_2 \in Y$, ..., $t_n \in D^{nm}$ such that $t_{n-1} \prec t_n$, $y_n \in Y$ so that, for all $1 \leqslant i \leqslant n$,

(i) $\|y_i - \sum_{j=(i-1)m+1}^{im} x_{t|_i}\| \leqslant 6\psi$,

(ii) $\|\sum_{j=(i-1)m+1}^{im} x_{t|_i}\| \leqslant \psi_1$,

(iii) $\|\sum_{i=1}^{n} \frac{y_i}{6\psi_1}\| \leqslant \phi$,

(iv) $\|\sum_{i=1}^{n} \frac{y_i - \sum_{j=(i-1)m+1}^{im} x_{t|_j}}{6\psi}\| \leqslant \phi_1$.

Then

$$\Big\|\sum_{i=1}^{mn} x_{t|_i}\Big\| \leqslant \Big\|\sum_{i=1}^{n}[y_i - \sum_{j=(i-1)m+1}^{im} x_{t|_j}]\Big\| + \Big\|\sum_{i=1}^{n} y_i\Big\|$$
$$\leqslant 6\psi\phi_1 + 6\psi_1\phi.$$

Since $\psi > \theta_m(X/Y)$, $\psi_1 > \theta_m(X)$, $\phi > \theta_n(Y)$, and $\phi_1 > \theta_n(X)$ were arbitrary, this will yield the inequality.

We now explain how to choose t_i and y_i. Assume that for some $k < n$, we have already chosen $t_1 \prec \ldots \prec t_k$, $t_i \in D^{im}$, and $y_1, \ldots, y_k$. Assume also that $(y_i/6\psi_1)_{i=1}^k$ and $((y_i - \sum_{j=(i-1)m+1}^{im} x_{t_k|_j})/6\psi)_{i=1}^k$ have been chosen by Player II against Player I, who is using strategies ϖ and ϖ_1, respectively. Let U, U_1 be the weak neighborhoods chosen for the next stage of the game by strategies ϖ and ϖ_1, respectively. By remark 1.2.3, we can assume that all the weak neighborhoods are convex.

By Claim 1, there exists a weak neighborhood W of 0 in X, which we can also assume is convex, such that if $x \in W \cap \psi_1 B_X$ satisfies $\|x\|_{X/Y} \leqslant \psi$, then there exists $y \in U \cap \frac{1}{2}U_1 \cap \psi_1 B_Y$ such that $\|y - x\| \leqslant 6\psi$. Let $Q : X \to X/Y$ denote the quotient map and, using the strategies χ and χ_1, choose $t_k \prec s_1 \prec \ldots \prec s_m = t_{k+1} \in D^{(k+1)m}$ such that for each $1 \leqslant j \leqslant m$,

$$x_{s_j} \in G_j \cap Q^{-1}(H_j) \cap \frac{1}{m}W \cap \frac{1}{2m}U_1.$$

Here, the sets H_j are determined by χ playing against Player II, chosing $x_{s_1} + X/Y, \ldots, x_{s_m} + X/Y$ and the sets G_j are determined by χ_1 playing against Player II's, chosing x_{s_1}, ...,

x_{s_m}. Note that $G_j \cap Q^{-1}(H_j)$ is a weak neighborhood of 0 in X. Playing according to χ and χ_1 guarantees that

$$\Big\| \sum_{j=km+1}^{(k+1)m} x_{t_{k+1}|_j} \Big\| = \Big\| \sum_{j=1}^{m} x_{s_j} \Big\| \leqslant \psi_1$$

and

$$\Big\| \sum_{j=km+1}^{(k+1)m} x_{t_{k+1}|_j} \Big\|_{X/Y} = \Big\| \sum_{j=1}^{m} x_{s_j} \Big\|_{X/Y} \leqslant \psi.$$

Since $\sum_{j=1}^{m} x_{s_j} \in \frac{1}{m} W + \ldots + \frac{1}{m} W = W$, there exists $y_{k+1} \in U \cap \frac{1}{2} U_1 \cap \psi_1 B_Y$ such that

$$\Big\| y_{k+1} - \sum_{j=1}^{m} x_{s|_j} \Big\| \leqslant 6\psi.$$

Note also that $y_{k+1} - \sum_{j=1}^{m} x_{s_j} \in \frac{1}{2} U_1 + \frac{1}{2m} U_1 + \ldots + \frac{1}{2m} U_1 = U_1$. Therefore

$$\frac{y_{k+1}}{\psi_1} \in U \cap B_Y \quad \text{and} \quad \frac{y_{k+1} - \sum_{j=km+1}^{(k+1)m} x_{t_{k+1}|_j}}{6\psi} \in U_1 \cap B_X$$

obey the rules coming from the winning strategies ϖ and ϖ_1, respectively. This completes the recursive choices. Items (i) and (ii) are seen to be satisfied from the construction, while items (iii) and (iv) follow from the fact that the outer sequences were chosen according to ϖ and ϖ_1.

□

3.1.3 A counterexample

For $p \in (1, \infty)$, contrary to P_p, none of the properties T_p, A_p, N_p is a three-space property. Before proving this result by giving a counterexample, let us introduce the following definition.

Definition 3.1.5. Let X be a Banach space and $p \in (1, \infty)$. We say that X has the *weak p-Banach-Saks property* if there exists a positive constant C such that for every weakly null sequence (x_n) in B_X and every $k \in \mathbb{N}$, we can find a subsequence $(x_{n_j})_j$ of (x_n) such that

$$\Big\| \sum_{j=1}^{k} x_{n_j} \Big\| \leq C k^{1/p}$$

for all $n_1 < \cdots < n_k$.

Let us notice that every Banach space with property N_p, $1 < p < \infty$, has the weak p-Banach-Saks property. For instance, use item (iii) of Theorem 1.2.11 and mimic the argument of $(iii) \Rightarrow (ii)$ in the proof of that statement.

Proposition 3.1.6. *Let $p \in (1, \infty)$. Then the properties T_p, A_p, and N_p are not three space properties.*

Proof. Let us consider the Kalton-Peck reflexive spaces Z_p (see [66] or [24]), that satisfies the following: Z_p may be normed in such a way that it has a closed subspace M isometric to ℓ_p with Z_p/M also isometric to ℓ_p. It is known that Z_p does not have the weak p-Banach-Saks property (see [24]). Hence, since ℓ_p has property T_p, we get the result by combining the previous remark and item (ii) of Theorem 1.2.13. □

In fact, we can even prove that Z_p does not have any of the concentration properties indexed by p considered in [40] and therefore deduce the following.

Proposition 3.1.7. *Let $p \in (1,\infty)$. Then the properties HFC_p, HIC_p, HC_p, $HFC_{p,d}$, and $HC_{p,d}$ introduced in Chapter 2 are not three spaces properties.*

Proof of Proposition 3.1.7. We use the notation from Theorem 6.1 [66]. For $k \in \mathbb{N}$, the map $g : \begin{cases} [\mathbb{N}]^k & \to & Z_p \\ \overline{n} & \mapsto & \sum_{j=1}^k u_{n_j} \end{cases}$ is 2-Lipschitz and satisfies

$$\|g(\overline{n}) - g(\overline{m})\| = 2^{1/p}\left(\frac{\ln(2k)}{p} + 1\right)k^{1/p}$$

for all integers $n_1 < m_1 < n_2 < \cdots < n_k < m_k$. Therefore, Z_p does not have any of the properties mentioned above. The result follows from the fact that ℓ_p has them all, which is due to Kalton and Randrianarivony [67]. □

Remark 3.1.8. Let $p \in (1,\infty)$. One can note the following more general result: if (P) is a property of Banach spaces satisfied by ℓ_p that implies the weak-p-Banach-Saks property, then (P) is not a three-space property. In particular, being asymptotic-ℓ_p or having asymptotic models dominated by ℓ_p are not three-space properties.

3.1.4 Asymptotic uniform flattenability

In the case $p = \infty$, the situation is different. First, we easily have the following.

Theorem 3.1.9. *The property T_∞ is a three-space property.*

Proof. Let us first recall that a separable Banach space is T_∞ if and only if it is isomorphic to a subspace of c_0 (see [45]). It was shown by Johnson and Zippin [61], who attributed it to Lindenstrauss, that being isomorphic to a subspace of c_0 is a three-space property. So we deduce from Theorem 1.2.15, Proposition 3.1.1 and Lemma 3.1.2 that T_∞ is a three-space property. □

By looking at the argument in [61], we can actually show slightly more.

Proposition 3.1.10. *Let $p \in (1,\infty]$ and B be any one of the properties T_p, A_p and N_p. Let X be a Banach space with a closed subspace Y such that Y has T_∞ and X/Y has B. Then X has B.*

Proof. By Theorem 1.2.15, we may assume that X is separable. Let $T : Y \to c_0$ be a linear embedding. It follows from Sobczyk's theorem that c_0 has the separable extension property. Therefore T extends to a bounded linear map $S : X \to c_0$. Define now $U : X \to c_0 \oplus X/Y$ by $Ux = (Sx, Qx)$ where $Q : X \to X/Y$ is the quotient map. It is then easy to check that U is a linear embedding from X into $c_0 \oplus X/Y$. Finally, since B passes clearly to direct sums, we deduce that $c_0 \oplus X/Y$ and therefore X have B. □

3.1.5 Summable Szlenk index

This subsection contains the proof of our main result on three-space properties. We will show that A_∞ is a three-space property.

Recall that a Banach space is said to have property A_∞ provided there exists a constant $c > 0$ such that for each $n \in \mathbb{N}$, Player I has a winning strategy in the $N(c, \infty, n)$ game (since $\mathsf{A}_\infty = \mathsf{N}_\infty$ according to Theorem 1.2.13): Player I chooses a weak neighborhood U_1 of 0 in X and Player II chooses $x_1 \in U_1 \cap B_X$. Player I chooses a weak neighborhood U_2 of 0 in X and Player II chooses $x_2 \in U_2 \cap B_X$. Play continues in this way until $x_1, \dots, x_n$ have been chosen. Player I wins if $\| \sum_{i=1}^n x_i \| \leqslant c$ and Player II wins otherwise.

As we are going to use a separable reduction, we will be able to use trees indexed by $\mathbb{N}$. For that purpose we let $T_n = \mathbb{N}^{\leq n}$.

It will be convenient for us to introduce the notions of *$\mathcal{U}$-weakly (or weak*) null* sequences or collections indexed by T_n in a Banach space, where $\mathcal{U}$ is a given free ultrafilter on $\mathbb{N}$. Let us assume, more generally, that $\mathcal{U}$ is a filter on $\mathbb{N}$. Given a Banach space X, we say that a sequence $(x_i)_{i=1}^\infty \subset X$ is *$\mathcal{U}$-weakly null* if it converges to 0 over $\mathcal{U}$ in the weak topology. The notion of *$\mathcal{U}$-weak*-null* for a sequence $(x_i^*)_{i=1}^\infty \subset X^*$ is defined similarly. A collection $(x_t)_{t \in T_n} \subset X$ is *$\mathcal{U}$-weakly null* provided that for each $t \in \{\varnothing\} \cup T_{n-1}$, $(x_{t \frown (m)})_{m=1}^\infty$ is $\mathcal{U}$-weakly null. We say $(x_t^*)_{t \in T_n} \subset X^*$ is *$\mathcal{U}$-weak*-null* provided that for each $t \in \{\varnothing\} \cup T_{n-1}$, $(x^*_{t \frown (m)})_{m=1}^\infty$ is $\mathcal{U}$-weak*-null. Note that for each Banach space X and each $n \in \mathbb{N}$, B_X admits a $\mathcal{U}$-weakly null collection and B_{X^*} admits a $\mathcal{U}$-weak*-null collection, namely the collections consisting entirely of zeros.

In the remainder of this section, we shall always assume that $\mathcal{U}$ is a free ultrafilter on $\mathbb{N}$.

Proposition 3.1.11. *Let Z be a Banach space such that Z^* is separable, and $(z_m^*)_{m=1}^\infty \subset Z$ a $\mathcal{U}$-weak*-null sequence in Z^*. For any $\delta > 0$, there exists a $\mathcal{U}$-weakly null sequence $(z_m)_{m=1}^\infty \subset B_Z$ such that*

$$\lim_{m \in \mathcal{U}} \operatorname{Re} \, z_m^*(z_m) \geqslant \lim_{m \in \mathcal{U}} \frac{\|z_m^*\|}{2} - \delta.$$

Proof. If $\lim_{m \in \mathcal{U}} \|z_m^*\| = 0$, simply take $z_m = 0$ for all $m \in \mathbb{N}$. Suppose that $r := \lim_{m \in \mathcal{U}} \|z_m^*\| > 0$. For each $m \in \mathbb{N}$, fix $x_m \in B_X$ such that $\operatorname{Re}\, z_m^*(x_m) > \|z_m^*\| - \delta$. Let $x^{**} = \text{weak}^*\text{-}\lim_{m \in \mathcal{U}} x_m$, where the limit is taken in $B_{X^{**}}$. Since X^* is separable, the weak*-topology on $B_{X^{**}}$ is metrizable, which means some subsequence $(u_m)_{m=1}^\infty$ of $(x_m)_{m=1}^\infty$ is weak*-convergent to x^{**}. Define $u_0 = 0$. For each $m \in \mathbb{N}$, let $C_m = \{i \in \mathbb{N}_0 : |z_m^*(u_i)| \leqslant \delta\}$, where $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. Note that $0 \in C_m$ for all $m \in \mathbb{N}$. Define $f : \mathbb{N} \to \mathbb{N}_0$ by letting $f(m) = \max C_m$ if $\max C_m < m$, and let $f(m) \in C_m \cap [m, \infty)$ be arbitrary if $\max C_m \geqslant m$. Note that $\lim_{m \in \mathcal{U}} f(m) = l \in \mathbb{N}_0 \cup \{\infty\}$, where $\mathbb{N}_0 \cup \{\infty\}$ is the one-point compactification of $\mathbb{N}_0$. We claim that $l = \infty$. Indeed, if $l < \infty$, then

$$f^{-1}(\{l\}) \cap \{m \in \mathbb{N} : |z_m^*(u_{l+1})| < \delta\} \in \mathcal{U},$$

and therefore there exists some $l < m_0 \in f^{-1}(\{l\}) \cap \{m \in \mathbb{N} : z_m^*(u_{l+1})\}$. But this means that $l + 1 \in C_{m_0}$ and $l = \max C_{m_0}$, this is a contradiction.

Define now $z_m = \frac{1}{2}(x_m - u_{f(m)}) \in B_X$. Since $\lim_{m \in \mathcal{U}} f(m) = \infty$ and $\text{weak}^*\text{-}\lim_{m \to \infty} u_m = x^{**}$, $\text{weak}^*\text{-}\lim_{m \in \mathcal{U}} u_{f(m)} = x^{**}$. Therefore $\text{weak-}\lim_{m \in \mathcal{U}} z_m = 0$. By our choice of $f(m)$, $|z_m^*(u_{f(m)})| \leqslant$

δ for all $m \in \mathbb{N}$, and

$$\lim_{m\in\mathcal{U}} \text{Re } z_m^*(z_m) \geqslant \lim_{m\in\mathcal{U}} \text{Re } \frac{1}{2} z_m^*(x_m) - \lim_{m\in\mathcal{U}} \frac{1}{2} |z_m^*(u_{f(m)})| \geqslant \frac{r}{2} - \delta.$$

□

We define $\alpha_n^{\mathcal{U}}(X)$ to be the infimum of $a \geqslant 0$ such that for any $\mathcal{U}$-weakly null $(x_t)_{t\in T_n} \subset B_X$,

$$\lim_{m_1\in\mathcal{U}} \dots \lim_{m_n\in\mathcal{U}} \Big\| \sum_{i=1}^n x_{(m_1,\dots,m_i)} \Big\| \leqslant a.$$

We define $\beta_n^{\mathcal{U}}(X)$ to be the infimum of $b > 0$ such that for any $\mathcal{U}$-weak*-null $(x_t^*)_{t\in T_n} \subset B_{X^*}$,

$$b \lim_{m_1\in\mathcal{U}} \dots \lim_{m_n\in\mathcal{U}} \Big\| \sum_{i=1}^n x^*_{(m_1,\dots,m_i)} \Big\| \geqslant \sum_{i=1}^n \lim_{m_1\in\mathcal{U}} \dots \lim_{m_i\in\mathcal{U}} \|x^*_{(m_1,\dots,m_i)}\|.$$

The next proposition details a characterization and a dual formulation of A_∞ for spaces with separable dual.

Proposition 3.1.12. *Let X be a Banach space such that X^* is separable.*

(i) X has A_∞ if and only if $\sup_n \alpha_n^{\mathcal{U}}(X) < \infty$.

(ii) For each $n \in \mathbb{N}$, $\alpha_n^{\mathcal{U}}(X) \leqslant 2\beta_n^{\mathcal{U}}(X)$.

(iii) For each $n \in \mathbb{N}$, $\beta_n^{\mathcal{U}}(X) \leqslant 2\alpha_n^{\mathcal{U}}(X)$.

Proof. (i) Since X^* is separable, there exists a metric d on B_X that is compatible with the weak topology. For each $n \in \mathbb{N}$, let $U_n = \{x \in X : d(x, 0) < 1/n\}$.

First assume that X does not have property A_∞. Then for each $a > 0$, there exists $n \in \mathbb{N}$ such that Player I fails to have a winning strategy in the $A(a, n)$ game. Since the $A(a, n)$ game is determined, Player II must have a winning strategy in the $A(a, n)$ game. We will choose $(x_t)_{t\in T_n}$ according to this winning strategy. First, let $x_{(m)} \in U_m \cap B_X$ be Player II's response if Player I opens the game with U_m. For $1 < k \leqslant n$ and $t = (m_1, \dots, m_k)$, let $x_t \in U_{m_k} \cap B_X$ be Player II's response if Players I and II have chosen $U_{m_1}, x_{(m_1)}, \dots, x_{(m_1,\dots,m_{k-1})}, U_{m_k}$. For each $t = (m_i)_{i=1}^n \in T_n$, since $U_{m_1}, x_{(m_1)}, U_{m_2}, x_{(m_1,m_2)}, \dots, U_{m_n}, x_{(m_1,\dots,m_n)}$ were chosen according to Player II's winning strategy, $\Big\| \sum_{i=1}^n x_{(m_1,\dots,m_i)} \Big\| > a$. Therefore

$$\lim_{m_1\in\mathcal{U}} \dots \lim_{m_n\in\mathcal{U}} \Big\| \sum_{i=1}^n x_{(m_1,\dots,m_i)} \Big\| \geqslant a.$$

Since $x_{t\frown(m)} \in U_m$ for each $t \in \{\varnothing\} \cup T_{n-1}$ and $m \in \mathbb{N}$, it follows that $(x_{t\frown(m)})_{m=1}^\infty$ is weakly null, and is therefore a $\mathcal{U}$-weakly null sequence, for each $t \in \{\varnothing\} \cup T_{n-1}$. Therefore this collection $(x_t)_{t\in T_n} \subset B_X$ witnesses the fact that $\alpha_n^{\mathcal{U}}(X) \geqslant a$. Since $a > 0$ was arbitrary, $\sup_n \alpha_n^{\mathcal{U}}(X) = \infty$. By contraposition, if $\sup_n \alpha_n^{\mathcal{U}}(X) < \infty$, X has property A_∞.

Next, suppose that X has property A_∞. Fix $a_0 > 0$ such that for all $n \in \mathbb{N}$, Player I has a winning strategy in the $A(a_0, n)$ game. Suppose that for some $n \in \mathbb{N}$, $\alpha_n^{\mathcal{U}}(X) > a_0$. Fix $\alpha_n^{\mathcal{U}}(X) > a > a_0$. There exists $(x_t)_{t\in T_n} \subset B_X$ which is $\mathcal{U}$-weakly null and such that

$$\lim_{m_1\in\mathcal{U}} \dots \lim_{m_n\in\mathcal{U}} \Big\| \sum_{i=1}^n x_{(m_1,\dots,m_i)} \Big\| > a.$$

Let V_1 be Player I's first choice according to a winning strategy in the $A(a_0, n)$ game. This means we can choose

$$m_1 \in \{m \in \mathbb{N} : x_{(m)} \in V_1\} \cap \{m \in \mathbb{N} : \|x_{(m)}\| > a\} \in \mathcal{U}, \quad \text{if } n = 1$$

and

$$\begin{aligned} &m_1 \in \{m \in \mathbb{N} : x_{(m)} \in V_1\} \\ &\cap \Big\{ m \in \mathbb{N} : \lim_{m_2\in\mathcal{U}} \dots \lim_{m_n\in\mathcal{U}} \Big\| x_{(m)} + \sum_{i=2}^n x_{(m,m_2,\dots,m_i)} \Big\| > a \Big\} \in \mathcal{U}, \quad \text{if } n > 1 \end{aligned}$$

Let Player II's choice in the $A(a_0, n)$ game be $x_{(m_1)}$. Next, assume that for some $1 \leqslant k < n$, $V_1, \dots, V_k$ and $m_1, \dots, m_k$ have been chosen such that

(a) $V_1, x_{(m_1)}, \dots, V_k, x_{(m_1,\dots,m_k)}$ have been chosen in the $A(a_0, n)$ game with Player I playing according to a winning strategy,

(b) we have the inequality

$$\lim_{m_{k+1}\in\mathcal{U}} \dots \lim_{m_n\in\mathcal{U}} \Big\| \sum_{i=1}^n x_{(m_1,\dots,m_i)} \Big\| > a,$$

We now describe the recursive step of the construction.
Assume first that $k + 1 < n$. Let V_{k+1} be Player I's next choice according to the winning strategy, let Player II's choice be $x_{(m_1,\dots,m_{k+1})}$, where

$$\begin{aligned} &m_{k+1} \in \{m \in \mathbb{N} : x_{(m_1,\dots,m_k,m)} \in V_{k+1}\} \cap \\ &\Big\{ m \in \mathbb{N} : \lim_{m_{k+2}\in\mathcal{U}} \dots \lim_{m_n\in\mathcal{U}} \Big\| x_{(m_1,\dots,m_k,m)} + \sum_{i\neq k+1; 1\leq i\leq n} x_{(m_1,\dots,m_i)} \Big\| > a \Big\} \in \mathcal{U} \end{aligned}$$

Assume, for the last step, that $k + 1 = n$. Let $V_{k+1} = V_n$ be Player I's next choice according to the winning strategy, let Player II's choice be $x_{(m_1,\dots,m_n)}$, where

$$\begin{aligned} &m_{k+1} = m_n \in \{m \in \mathbb{N} : x_{(m_1,\dots,m_k,m)} \in V_n\} \cap \\ &\Big\{ m \in \mathbb{N} : \Big\| x_{(m_1,\dots,m_k,m)} + \sum_{i=1}^k x_{(m_1,\dots,m_i)} \Big\| > a \Big\} \in \mathcal{U} \end{aligned}$$

This completes the recursive construction, from which we it follows that $\|\sum_{i=1}^n x_{(m_1,\dots,m_i)}\| > a$. However, since $V_1, x_{(m_1)}, \dots, V_n, x_{(m_1,\dots,m_n)}$ were chosen with Player I playing according to a winning strategy in the $A(a_0, n)$ game, $\|\sum_{i=1}^n x_{(m_1,\dots,m_i)}\| \leqslant a_0$. Since $a_0 < a$, this is a contradiction. Hence $\sup_n \alpha_n^{\mathcal{U}}(X) \leqslant a_0 < \infty$. Therefore if X has property A_∞, $\sup_n \alpha_n^{\mathcal{U}}(X) < \infty$.

(ii) Fix $a < \alpha_n^{\mathcal{U}}(X)$. Then there exists a $\mathcal{U}$-weakly null collection $(x_t)_{t\in T_n} \subset B_X$ such that

$$a < \lim_{m_1\in\mathcal{U}} \dots \lim_{m_n\in\mathcal{U}} \Big\| \sum_{i=1}^n x_{(m_1,\dots,m_i)} \Big\|.$$

For each $t = (m_1,\dots,m_n) \in \mathbb{N}^n$, choose $y_t^* \in \frac{1}{2}B_{X^*}$ such that

$$\operatorname{Re}\, y_t^*\Big(\sum_{i=1}^n x_{(m_1,\dots,m_i)}\Big) = \frac{1}{2}\Big\|\sum_{i=1}^n x_{(m_1,\dots,m_i)}\Big\|.$$

Then

$$\frac{a}{2} < \lim_{m_1\in\mathcal{U}} \dots \lim_{m_n\in\mathcal{U}} \operatorname{Re}\, y^*_{(m_1,\dots,m_n)}\Big(\sum_{i=1}^n x_{(m_1,\dots,m_i)}\Big).$$

For $t \in \mathbb{N}^n$, set $z_t^* = y_t^*$ and for each $t \in \{\varnothing\} \cup T_{n-1}$, define

$$z_t^* = \lim_{m_{|t|+1}\in\mathcal{U}} \dots \lim_{m_n\in\mathcal{U}} y^*_{t\frown(m_{|t|+1},\dots,m_n)},$$

where all limits are taken with respect to the weak*-topology. For $(m_1,\dots,m_k) \in T_n$, define $x^*_{(m_1,\dots,m_k)} = z^*_{(m_1,\dots,m_k)} - z^*_{(m_1,\dots,m_{k-1})} \in B_{X^*}$. Note that $(x_t^*)_{t\in T_n} \subset B_{X^*}$ is $\mathcal{U}$-weak*-null, which implies that

$$\begin{aligned}\sum_{i=1}^n \lim_{m_1\in\mathcal{U}} \dots \lim_{m_i\in\mathcal{U}} \|x^*_{(m_1,\dots,m_i)}\| &\leqslant \beta_n^{\mathcal{U}}(X) \lim_{m_1\in\mathcal{U}} \dots \lim_{m_n\in\mathcal{U}} \Big\|\sum_{i=1}^n x^*_{(m_1,\dots,m_i)}\Big\| \quad (*)\\ &= \beta_n^{\mathcal{U}}(X) \lim_{m_1\in\mathcal{U}} \dots \lim_{m_n\in\mathcal{U}} \|z^*_{(m_1,\dots,m_n)} - z^*_\varnothing\|\\ &\leqslant \beta_n^{\mathcal{U}}(X).\end{aligned}$$

Note that because $(x_t)_{t\in T_n}$ is $\mathcal{U}$-weakly null and $(x_t^*)_{t\in T_n}$ is $\mathcal{U}$-weak*-null, it holds that for distinct $i,j \in \{1,\dots,n\}$,

$$\lim_{m_1\in\mathcal{U}} \dots \lim_{m_n\in\mathcal{U}} x^*_{(m_1,\dots,m_i)}(x_{(m_1,\dots,m_j)}) = 0.$$

Similarly, for each $1 \leqslant i \leqslant n$, $\lim_{m_1\in\mathcal{U}} \dots \lim_{m_n\in\mathcal{U}} z^*_\varnothing(x_{(m_1,\dots,m_i)}) = 0$.

Combining the facts above, we can write

$$\begin{aligned}
\frac{a}{2} &< \lim_{m_1\in\mathcal{U}}\dots\lim_{m_n\in\mathcal{U}} \mathrm{Re}\ z^*_{(m_1,\dots,m_n)}\Big(\sum_{i=1}^n x_{(m_1,\dots,m_i)}\Big)\\
&= \lim_{m_1\in\mathcal{U}}\dots\lim_{m_n\in\mathcal{U}} \mathrm{Re}\ (z^*_{(m_1,\dots,m_n)} - z^*_{\varnothing})\Big(\sum_{i=1}^n x_{(m_1,\dots,m_i)}\Big)\\
&= \lim_{m_1\in\mathcal{U}}\dots\lim_{m_n\in\mathcal{U}} \mathrm{Re}\ \Big(\sum_{i=1}^n x^*_{(m_1,\dots,m_i)}\Big)\Big(\sum_{i=1}^n x_{(m_1,\dots,m_i)}\Big)\\
&= \sum_{i=1}^n \lim_{m_1\in\mathcal{U}}\dots\lim_{m_n\in\mathcal{U}} \mathrm{Re}\ x^*_{(m_1,\dots,m_i)}(x_{(m_1,\dots,m_i)})\\
&= \sum_{i=1}^n \lim_{m_1\in\mathcal{U}}\dots\lim_{m_i\in\mathcal{U}} \mathrm{Re}\ x^*_{(m_1,\dots,m_i)}(x_{(m_1,\dots,m_i)})\\
&\leqslant \sum_{i=1}^n \lim_{m_1\in\mathcal{U}}\dots\lim_{m_i\in\mathcal{U}} \|x^*_{(m_1,\dots,m_i)}\| \leqslant \beta_n^{\mathcal{U}}(X).
\end{aligned}$$

Since $a < \alpha_n^{\mathcal{U}}(X)$ was arbitrary, we are done.

(iii) Fix $b < \beta_n^{\mathcal{U}}(X)$ and $\delta > 0$. Then there exists a collection $(x_t^*)_{t\in T_n} \subset B_{X^*}$ which is $\mathcal{U}$-weak*-null and

$$b \lim_{m_1\in\mathcal{U}}\dots\lim_{m_n\in\mathcal{U}}\Big\|\sum_{i=1}^n x^*_{(m_1,\dots,m_i)}\Big\| < \sum_{i=1}^n \lim_{m_1\in\mathcal{U}}\dots\lim_{m_i\in\mathcal{U}}\|x^*_{(m_1,\dots,m_i)}\|.$$

Note that this inequality and $(*)$ imply that

$$\lim_{m_1\in\mathcal{U}}\dots\lim_{m_n\in\mathcal{U}}\Big\|\sum_{i=1}^n x^*_{(m_1,\dots,m_i)}\Big\| > 0.$$

We define a collection $(x_t)_{t\in T_n} \subset B_X$ which is $\mathcal{U}$-weakly null and such that for each $1 \leqslant i \leqslant n$,

$$\lim_{m_1\in\mathcal{U}}\dots\lim_{m_i\in\mathcal{U}} \mathrm{Re}\ x^*_{(m_1,\dots,m_i)}(x_{(m_1,\dots,m_i)}) \geqslant \frac{1}{2}\lim_{m_1\in\mathcal{U}}\dots\lim_{m_i\in\mathcal{U}}\|x^*_{(m_1,\dots,m_i)}\| - \delta.$$

To that end, for each $t \in \{\varnothing\}\cup T_{n-1}$, choose, as it is allowed by Proposition 3.1.11, $(x_{t\frown(m)})_{m=1}^\infty \subset B_X$ to be a $\mathcal{U}$-weakly null sequence such that

$$\lim_{m\in\mathcal{U}} \mathrm{Re}\ x^*_{t\frown(m)}(x_{t\frown(m)}) \geqslant \frac{1}{2}\lim_{m\in\mathcal{U}}\|x^*_{t\frown(m)}\| - \delta.$$

Note that, since $(x_t)_{t\in T_n}$ is $\mathcal{U}$-weakly null and $(x_t^*)_{t\in T_n}$ is $\mathcal{U}$-weak*-null, we have again that for all $1 \leq i \neq j \leq n$

$$\lim_{m_1\in\mathcal{U}}\dots\lim_{m_n\in\mathcal{U}} x^*_{(m_1,\dots,m_i)}(x_{(m_1,\dots,m_j)}) = 0.$$

Then, we can write

$$\begin{aligned}
&\frac{b}{2}\lim_{m_1\in\mathcal{U}}\dots\lim_{m_n\in\mathcal{U}}\Big\|\sum_{i=1}^n x^*_{(m_1,\dots,m_i)}\Big\| - \delta n \\
&< \frac{1}{2}\sum_{i=1}^n \lim_{m_1\in\mathcal{U}}\dots\lim_{m_i\in\mathcal{U}}\|x^*_{(m_1,\dots,m_i)}\| - \delta n \\
&\leqslant \sum_{i=1}^n \lim_{m_1\in\mathcal{U}}\dots\lim_{m_i\in\mathcal{U}} \mathrm{Re}\; x^*_{(m_1,\dots,m_i)}(x_{(m_1,\dots,m_i)}) \\
&= \lim_{m_1\in\mathcal{U}}\dots\lim_{m_n\in\mathcal{U}} \mathrm{Re}\;\Big(\sum_{i=1}^n x^*_{(m_1,\dots,m_i)}\Big)\Big(\sum_{i=1}^n x_{(m_1,\dots,m_i)}\Big) \\
&\leqslant \lim_{m_1\in\mathcal{U}}\dots\lim_{m_n\in\mathcal{U}}\Big\|\sum_{i=1}^n x^*_{(m_1,\dots,m_i)}\Big\|\Big\|\sum_{i=1}^n x_{(m_1,\dots,m_i)}\Big\| \\
&\leqslant \alpha_n^{\mathcal{U}}(X)\lim_{m_1\in\mathcal{U}}\dots\lim_{m_n\in\mathcal{U}}\Big\|\sum_{i=1}^n x^*_{(m_1,\dots,m_i)}\Big\|.
\end{aligned}$$

Since $\lim_{m_1\in\mathcal{U}}\dots\lim_{m_n\in\mathcal{U}}\Big\|\sum_{i=1}^n x^*_{(m_1,\dots,m_i)}\Big\| > 0$, and since $\delta > 0$ and $b < \beta_n^{\mathcal{U}}(X)$ were arbitrary, we are done.

□

Remark 3.1.13. In item (i) of the preceding proof, we actually showed that if X^* is separable, then for each $n \in \mathbb{N}$, $\alpha_n^{\mathcal{U}}(X)$ is the infimum of $a > 0$ such that Player I has a winning strategy in the $A(a, n)$ game.

We can now turn to the heart of the proof.

Lemma 3.1.14. *For any Banach space X with X^* separable and any subspace Y of X,*

$$\alpha_n^{\mathcal{U}}(X) \leqslant 40\max\{\alpha_n^{\mathcal{U}}(Y), \alpha_n^{\mathcal{U}}(X/Y)\}^2.$$

Proof. If X is finite-dimensional, then $\alpha_n^{\mathcal{U}}(X) = \alpha_n^{\mathcal{U}}(Y) = \alpha_n(X/Y) = 0$, so assume X is infinite-dimensional. In this case, at least one of Y, X/Y must also be infinite-dimensional, which means

$$b := \max\{\beta_n^{\mathcal{U}}(Y), \beta_n^{\mathcal{U}}(X/Y)\} \geqslant 1.$$

Fix $(x_t^*)_{t\in T_n} \subset B_{X^*}$ $\mathcal{U}$-weak*-null. We will define a bounded, $\mathcal{U}$-weak*-null collection $(y_t^*)_{t\in T_n} \subset Y^\perp$ such that for each $1 \leqslant i \leqslant n$,

$$\lim_{m_1\in\mathcal{U}}\dots\lim_{m_i\in\mathcal{U}}\|x^*_{(m_1,\dots,m_i)} - y^*_{(m_1,\dots,m_i)}\| \leqslant 2\lim_{m_1\in\mathcal{U}}\dots\lim_{m_i\in\mathcal{U}}\|x^*_{(m_1,\dots,m_i)}\|_{X^*/Y^\perp}.$$

To that end, for each $t = (m_1, \dots, m_i) \in T_n$, fix $w_t^* \in Y^\perp$ such that

$$\|x_t^* - w_t^*\| < \|x_t^*\|_{X^*/Y^\perp} + 2^{-m_i}$$

and note that $w_t^* \in 3B_{Y^\perp}$. For $t \in \{\varnothing\} \cup T_{n-1}$, let $v_t^* = \text{weak}^*\text{-}\lim_{m\in\mathcal{U}} w^*_{t\frown(m)} \in 3B_{Y^\perp}$ and let $y^*_{t\frown(m)} = w^*_{t\frown(m)} - v_t^*$. It is clear that $(y_t^*)_{t\in T_n} \subset Y^\perp$ is bounded and $\mathcal{U}$-weak*-null. Note that for any $t \in \{\varnothing\} \cup T_{n-1}$, weak*-$\lim_{m\in\mathcal{U}}(w^*_{t\frown(m)} - x^*_{t\frown(m)}) = v^*_t - 0 = v^*_t$. By weak*-lower semicontinuity of the norm, it follows that

$$\|v^*_t\| \leqslant \lim_{m\in\mathcal{U}}\|w^*_{t\frown(m)} - x^*_{t\frown(m)}\| = \lim_{m\in\mathcal{U}}\|x^*_{t\frown(m)}\|_{X^*/Y^\perp}.$$

Therefore

$$\lim_{m\in\mathcal{U}}\|x^*_{t\frown(m)} - y^*_{t\frown(m)}\| \leqslant \lim_{m\in\mathcal{U}}\|x^*_{t\frown(m)} - w^*_{t\frown(m)}\| + \|v^*_t\| \leqslant 2\lim_{m\in\mathcal{U}}\|x^*_{t\frown(m)}\|_{X^*/Y^\perp}.$$

From this it follows that

$$\lim_{m_1\in\mathcal{U}}\dots\lim_{m_i\in\mathcal{U}}\|x^*_{(m_1,\dots,m_i)} - y^*_{(m_1,\dots,m_i)}\| \leqslant 2\lim_{m_1\in\mathcal{U}}\dots\lim_{m_i\in\mathcal{U}}\|x^*_{(m_1,\dots,m_i)}\|_{X^*/Y^\perp}.$$

Since $(y^*_t)_{t\in T_n} \subset Y^\perp = (X/Y)^*$ is $\mathcal{U}$-weak*-null and bounded, by homogeneity, we have that

$$b\lim_{m_1\in\mathcal{U}}\dots\lim_{m_n\in\mathcal{U}}\Big\|\sum_{i=1}^n y^*_{(m_1,\dots,m_i)}\Big\| \geqslant \sum_{i=1}^n \lim_{m_1\in\mathcal{U}}\dots\lim_{m_i\in\mathcal{U}}\|y^*_{(m_1,\dots,m_i)}\|.$$

Let us prove that

$$b\lim_{m_1\in\mathcal{U}}\dots\lim_{m_n\in\mathcal{U}}\Big\|\sum_{i=1}^n x^*_{(m_1,\dots,m_i)}\Big\| \geqslant \frac{1}{1+4b}\sum_{i=1}^n \lim_{m_1\in\mathcal{U}}\dots\lim_{m_i\in\mathcal{U}}\|x^*_{(m_1,\dots,m_i)}\|.$$

We start with the easy case and suppose first that

$$\sum_{i=1}^n \lim_{m_1\in\mathcal{U}}\dots\lim_{m_i\in\mathcal{U}}\|x^*_{(m_1,\dots,m_i)}\|_{X^*/Y^\perp} \geqslant \frac{1}{1+4b}\sum_{i=1}^n \lim_{m_1\in\mathcal{U}}\dots\lim_{m_i\in\mathcal{U}}\|x^*_{(m_1,\dots,m_i)}\|.$$

Since $(x^*_t|_Y)_{t\in T_n} \subset B_{Y^*} = B_{X^*/Y^\perp}$ is $\mathcal{U}$-weak*-null,

$$\begin{aligned} b\lim_{m_1\in\mathcal{U}}\dots\lim_{m_n\in\mathcal{U}}\Big\|\sum_{i=1}^n x^*_{(m_1,\dots,m_i)}\Big\| &\geqslant b\lim_{m_1\in\mathcal{U}}\dots\lim_{m_n\in\mathcal{U}}\Big\|\sum_{i=1}^n x^*_{(m_1,\dots,m_i)}\Big\|_{X^*/Y^\perp} \\ &\geqslant \sum_{i=1}^n \lim_{m_1\in\mathcal{U}}\dots\lim_{m_i\in\mathcal{U}}\|x^*_{(m_1,\dots,m_i)}\|_{X^*/Y^\perp} \\ &\geqslant \frac{1}{1+4b}\sum_{i=1}^n \lim_{m_1\in\mathcal{U}}\dots\lim_{m_i\in\mathcal{U}}\|x^*_{(m_1,\dots,m_i)}\|. \end{aligned}$$

Next suppose that

$$\sum_{i=1}^n \lim_{m_1\in\mathcal{U}}\dots\lim_{m_i\in\mathcal{U}}\|x^*_{(m_1,\dots,m_i)}\|_{X^*/Y^\perp} < \frac{1}{1+4b}\sum_{i=1}^n \lim_{m_1\in\mathcal{U}}\dots\lim_{m_i\in\mathcal{U}}\|x^*_{(m_1,\dots,m_i)}\|.$$

Then since $b \geqslant 1$,

$$\begin{aligned}
& b \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_n \in \mathcal{U}} \Big\| \sum_{i=1}^{n} x^*_{(m_1,\dots,m_i)} \Big\| \geqslant b \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_n \in \mathcal{U}} \Big\| \sum_{i=1}^{n} y^*_{(m_1,\dots,m_i)} \Big\| \\
& - b \sum_{i=1}^{n} \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \| x^*_{(m_1,\dots,m_i)} - y^*_{(m_1,\dots,m_i)} \| \\
& \geqslant \sum_{i=1}^{n} \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \| y^*_{(m_1,\dots,m_i)} \| - 2b \sum_{i=1}^{n} \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \| x^*_{(m_1,\dots,m_i)} \|_{X^*/Y^\perp} \\
& \geqslant \sum_{i=1}^{n} \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \| x^*_{(m_1,\dots,m_i)} \| - \sum_{i=1}^{n} \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \| x^*_{(m_1,\dots,m_i)} - y^*_{(m_1,\dots,m_i)} \| \\
& - 2b \sum_{i=1}^{n} \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \| x^*_{(m_1,\dots,m_i)} \|_{X^*/Y^\perp} \\
& \geqslant \sum_{i=1}^{n} \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \| x^*_{(m_1,\dots,m_i)} \| - 4b \sum_{i=1}^{n} \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \| x^*_{(m_1,\dots,m_i)} \|_{X^*/Y^\perp} \\
& \geqslant \sum_{i=1}^{n} \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \| x^*_{(m_1,\dots,m_i)} \| - \frac{4b}{1+4b} \sum_{i=1}^{n} \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \| x^*_{(m_1,\dots,m_i)} \| \\
& = \frac{1}{1+4b} \sum_{i=1}^{n} \lim_{m_1 \in \mathcal{U}} \dots \lim_{m_i \in \mathcal{U}} \| x^*_{(m_1,\dots,m_i)} \|.
\end{aligned}$$

Combining the two previous paragraphs we get

$$\beta_n^{\mathcal{U}}(X) \leqslant b(1+4b) \leqslant 5b^2 = 5 \max\{\beta_n^{\mathcal{U}}(Y), \beta_n^{\mathcal{U}}(X/Y)\}^2.$$

Combining this inequality with items (ii) and (iii) of Proposition 3.1.12 yields

$$\alpha_n^{\mathcal{U}}(X) \leqslant 40 \max\{\alpha_n^{\mathcal{U}}(Y), \alpha_n^{\mathcal{U}}(X/Y)\}^2.$$

□

We can now state and prove our result.

Theorem 3.1.15. *The property* A_∞ *is a three-space property.*

Proof. Assume first that Y is a closed subspace of a Banach space X such that Y and X/Y are in $\mathsf{A}_\infty \cap \mathsf{Sep}$. Then Y and X/Y are separable Asplund spaces and $Y^* = X^*/Y^\perp$ and $(X/Y)^* = Y^\perp$ are separable. So X^* is separable and we can apply Lemma 3.1.14 and item (i) of Proposition 3.1.12 to deduce that X has A_∞. We have shown that membership in $\mathsf{A}_\infty \cap \mathsf{Sep}$ is a 3SP. It then follows from Theorem 1.2.15 and Lemma 3.1.2 that A_∞ is a 3SP.

□

Remark 3.1.16. Since reflexivity is also a three-space property (cf [69]), we can deduce that property HFC_∞ defined in Chapter 2, which is equivalent to being reflexive and asymptotic-c_0, is a three-space property. However, the question of whether or not property HC_∞ is a three-space property remains open.

3.2 Non-linear stabilities and examples

As one of the main applications of Chapter 1, we will exhibit two new properties related to the asymptotic smoothness of Banach spaces, that are stable under coarse Lipschitz equivalences

3.2.1 Results of non-linear stability

The following results were obtained by Godefroy, Kalton, and Lancien in [45] and [46].

Theorem 3.2.1. *Let $p \in (1, \infty]$.*

1. *The class T_p is stable under Lipschitz equivalences.*
2. *The class P_p is stable under coarse Lipschitz equivalences.*
3. *The class $\mathsf{A}_\infty = \mathsf{N}_\infty$ is stable under coarse Lipschitz equivalences.*

In fact, statements (2) and (3) are only proved for uniform homeomorphisms in [46] in the separable case. The adaptation for coarse Lipschitz equivalences relies on characterization (*iii*) in Proposition 1.1.4, which allows to apply the so-called Gorelik principle (see also [32] for details). Then the non-separable case can easily be deduced by a standard separable saturation argument combined with the separable determination of these properties. It is then natural to wonder about the non-linear stability of the classes A_p and N_p for $1 < p < \infty$. The results we have detailed in Section 1.2.2 together with a careful examination of the statements in [46] or [32] will allow us to easily obtain strong new stability results. We start with the following.

Theorem 3.2.2. *For any $p \in (1, \infty)$, the class A_p is stable under coarse Lipschitz equivalences.*

Proof. Let $X \in \mathsf{A}_p$ and Y a Banach space such that there exists a coarse Lipschitz equivalence f from X to Y. Then, Corollary 6.7 in [32], which is an extension of results in [46], insures the existence of a universal constant $K > 0$ and a constant $M > 0$ (depending on f) so that for any $\varepsilon > 0$, there exists a norm $|\ |$ on Y satisfying

$$\forall y \in Y,\ \|y\|_Y \leq |y| \leq M\|y\|_Y \ \text{ and } \ \forall \sigma \in [0,1],\ \overline{\rho}_{|\ |}(KM^2\sigma) \leq \overline{\rho}_X(\sigma) + \varepsilon.$$

With this result at hand, it is clear that characterization (*iii*) of A_p in Theorem 1.2.10 is stable under coarse Lipschitz equivalences. □

We also have.

Theorem 3.2.3. *For any $p \in (1, \infty)$, the class N_p is stable under coarse-Lipschitz equivalences.*

Proof. Similarly to the previous proof, this is a direct consequence of Corollary 6.7 in [32] and characterization (*iii*) of N_p in Theorem 1.2.11. □

Obviously the above argument can also be applied to prove that A_∞ is stable under coarse Lipschitz equivalences, which, as we explained, was already known.

Problem 3.2.4. In [64], N. Kalton proved that for $1 < p < \infty$, the class T_p is not stable under uniform homeomorphisms. It is not known however whether the class T_∞ is stable under coarse Lipschitz isomorphisms (or even uniform homeomorphisms). In fact, a positive answer would imply that a Banach space coarse Lipschitz equivalent to c_0 is linearly isomorphic to c_0, which is an important open question. Indeed, it is known that the class of all $\mathcal{L}_\infty$ spaces is stable under coarse Lipschitz equivalences [52] and that a $\mathcal{L}_\infty$ subspace of c_0 is isomorphic to c_0 [60].

3.2.2 Examples

To conclude this chapter, we gather a few known examples of T_∞ or A_∞ spaces and related problems.

Examples: non-separable uniformly flattenable spaces

The first obvious examples of non-separable T_∞ (or equivalently, AUF-renormable) spaces are given by $c_0(\Gamma)$ spaces, with Γ uncountable.

Proposition 3.2.5. *For any set Γ, the space $c_0(\Gamma)$ equipped with its natural norm is AUF.*

Proof. It follows immediately from the definition of the norm of $c_0(\Gamma)$ that

$$\forall t \in (0,1) \;\; \overline{\rho}_{c_0(\Gamma)}(t) = 0.$$

□

The next result was already known (see the remark after the proof). We present first a proof using that T_∞ is a 3SP.

Theorem 3.2.6. *Let K be a compact scattered space such that its Cantor derived set of order ω, $K^{(\omega)}$ is empty. Then $C(K)$ is T_∞.*

Proof. We shall prove it by induction on $n \in \mathbb{N}$ such that $K^{(n)} = \emptyset$. If $n = 1$, then $K' = \emptyset$ and K is finite. Therefore $C(K)$ is finite-dimensional and thus is T_∞. Assume that the statement is true for $n \in \mathbb{N}$ and that $K^{(n+1)} = \emptyset$. The subspace of $C(K)$ defined by $Y = \{f \in C(K),\ f_{|_{K'}} = 0\}$ is clearly isometric to $c_0(K \setminus K')$ and by Proposition 3.2.5 is T_∞. Let now Q be the restriction mapping from $C(K)$ to $C(K')$. It follows from the Tietze extension theorem that Q is onto. Since Y is the kernel of Q, we have that $C(K')$ is isomorphic to $C(K)/Y$. By induction hypothesis, $C(K')$ and thus $C(K)/Y$ are T_∞. It now follows from Theorem 3.1.9 that $C(K)$ is T_∞. □

Remark 3.2.7. As we already mentioned, this is not a new result. Let us indicate a few other ways to prove it.

1. Let K be a compact space such that $K^{(n)} = \emptyset$, $n \in \mathbb{N}$. The dual of $C(K)$ is isometric to $\ell_1(K)$. Define the following equivalent norm on $\ell_1(K)$:

$$\forall \mu \in \ell_1(K), \quad |\mu| = \sum_{x \in K} \alpha_x |\mu(x)|,$$

where $\alpha_x = 2^{-i}$ with $0 \leq i \leq n-1$ such that $x \in K^{(i)} \setminus K^{(i+1)}$. This formula comes from [70], where it is proved that this norm is 1-AUC* and is the dual norm of an equivalent norm on $C(K)$. So its predual norm is AUF.

2. Let X be a separable subspace of $C(K)$ and denote Y the closed sub-$*$-algebra of $C(K)$ generated by X. Then Y is isometric to a space $C(L)$, where L is a compact metrizable space such that $L^{(\omega)} = \emptyset$. It follows from [15] that Y is either finite dimensional or isomorphic to $c_0(\mathbb{N})$. So X is T_∞ and we can apply the separable determination of T_∞ (Theorem 1.2.15) to deduce that $C(K)$ is T_∞.

3. We conclude with the most sophisticated argument. It is known that if K is a compact space such that $K^{(\omega)} = \emptyset$, then $C(K)$ is Lipschitz isomorphic to some $c_0(\Gamma)$ (see [33]). On the other hand, being AUF renormable is stable under Lipschitz isomorphisms ([45] for the separable case and [32] for the general case, or use separable determination and saturation).

It is also important to mention that Theorem 3.2.6 provides (only in the non-separable setting) examples of T_∞ spaces that are not isomorphic to a quotient or a subspace of a $c_0(\Gamma)$ space. Indeed we have

Theorem 3.2.8. *There exists a compact space K such that $K^{(3)} = \emptyset$, but $C(K)$ is not isomorphic to a quotient of a subspace of a $c_0(\Gamma)$ space.*

Let us indicate this now classical construction. There exists a scattered separable uncountable compact space K so that $K^{(3)} = \emptyset$. This space is often called the Mrówka-Isbell space and its construction is due to Mrówka [80] and Isbell (credited in [44]). We also refer to its description in [102] page 1757, where its construction is based on the Johnson-Lindenstrauss space JL_0 [58]. Since K is separable, $C(K)$ admits a countable family of separating functionals (the Dirac maps at the points of a dense countable subset of K). But $C(K)$ is not separable, as K is uncountable and scattered and therefore non metrizable. It follows that $C(K)$ is not weakly Lindelöf determined (WLD in short): see Theorem 5.37 and Proposition 5.40 in [50], or see [100]. We conclude by recalling that $c_0(\Gamma)$ is always WLD and that being WLD is stable by passing to subspaces or quotients (see also [50] and references therein).

Problem 3.2.9. We do not know if there exists a T_∞ space which is not isomorphic to quotient of a subspace of a $C(K)$ space with $K^{(\omega)} = \emptyset$.

An interesting A_∞ space

We already explained that showing that T_∞ is stable under coarse Lipschitz equivalences would imply that a Banach space coarse Lipschitz equivalent to c_0 is linearly isomorphic to c_0. At this point it is only known that a Banach space coarse Lipschitz equivalent to c_0 is A_∞ and $\mathcal{L}_\infty$. Another hope was to show that a separable Banach space which is A_∞ and $\mathcal{L}_\infty$ is necessarily T_∞ (see conjecture after Theorem 5.6 in [46]). Let us mention here that this question has been solved negatively by Argyros, Gasparis, and Motakis in [4], who showed the existence of a separable Banach space X which is A_∞ and $\mathcal{L}_\infty$ but so that every infinite dimensional subspace of X contains an infinite dimensional reflexive subspace.

Chapter 4

Some asymptotic types

This is a joint work with Florent Baudier. The main goal of this chapter is to prove an asymptotic version of the following result.

Theorem 4.0.1 (Pisier, [91] and [90]). *The following assertions are equivalent:*
(i) ℓ_1 is not finitely representable in X;
(ii) ℓ_1 is not crudely finitely representable in X;
(iii) X is B-convex;
(iv) X has infratype p for some $p > 1$;
(v) X has Rademacher type p for some $p > 1$;
(vi) X has stable type p for some $p > 1$;
(vii) X has stable type 1.

We chose to present the proof by following the papers by Pisier, [90] and [91]. For sake of completeness, we include all the proofs, even though some of them are basically the same as Pisier's.

We will finish this chapter with another shorter proof of the fact that a p-uniformly smooth Banach space has Enflo type p, which is inspired by the proof of the concentration inequality from [67].

4.1 An overview of the local results

All Banach spaces in these notes are assumed to be real and infinite-dimensional unless otherwise stated. We denote the closed unit ball of a Banach space X by B_X, and its unit sphere by S_X. Given a Banach space X with norm $\|\cdot\|_X$, we simply write $\|\cdot\|$ as long as it is clear from the context on which space it is defined.

First introduced by Beck in 1962 (see [14]) to characterize the Banach spaces for which a certain strong law of large numbers holds, B-convexity was defined as follows: a Banach space X is B-convex if there exist $n \in \mathbb{N}$ and $\varepsilon > 0$ such that for every choice $x_1, \ldots, x_n$ of elements in B_X, at least one choice of signs gives

$$\Big\| \sum_{i=1}^{n} \pm x_i \Big\| \leq n(1-\varepsilon).$$

If such $n \in \mathbb{N}$ and $\varepsilon > 0$ exist, we say that X is (n, ε) convex.
Soon after, Giesy proved that this property is equivalent to not containing the ℓ_1^n's uniformly (we say that a normed linear space X contains the ℓ_1^n's uniformly if there exists a constant $\lambda > 0$ such that, for all $n \in \mathbb{N}$, one can find $x_1, \ldots, x_n \in S_X$ so that $\lambda^{-1}\|a\|_{\ell_1^n} \leq \|\sum_{i=1}^n a_i x_i\| \leq \lambda \|a\|_{\ell_1^n}$ for every $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$) and its stability under linear isomorphism and under passing to quotients and duals ([43]). He also proved that a B-convex space with an unconditional basis is reflexive, leaving open the question of this implication for any B-convex space. This question was answered negatively by James in 1973 ([57]).
Next, Pisier gave another proof in [91] of the characterization of B-convexity by not containing the ℓ_1^n's uniformly or having Rademacher type $p \in (1, 2]$, where a normed linear space X is said to have Rademacher type $p \in (0, 2]$ if one can find a constant C such that for every finite family $(x_i)_{i=1}^n$ in X, we have

$$\frac{1}{2^n} \sum_{(\varepsilon_i)_{i=1}^n \in \{\pm 1\}^n} \Big\| \sum_{i=1}^n \varepsilon_i x_i \Big\| \leq C \Big(\sum_{i=1}^n \|x_i\|^p \Big)^{1/p}.$$

To do so, he developed the submultiplicativity method, that has been used many times since then.
This method lead to results involving different notions of types. Before summarizing some of them in one statement, we recall two definitions.

First, we say that a normed linear space X has infratype $p \in (0, 2]$ if one can find a constant C such that for every finite family $(x_i)_{i=1}^n$ in X, we have

$$\inf_{(\varepsilon_i)_{i=1}^n \in \{\pm 1\}^n} \Big\| \sum_{i=1}^n \varepsilon_i x_i \Big\| \leq C \Big(\sum_{i=1}^n \|x_i\|^p \Big)^{1/p}.$$

Secondly, if $p \in (0, 2]$ and if $(\xi_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d random variables on a probability space $(\Omega, \mathcal{P})$ with characteristic function $t \in \mathbb{R} \mapsto e^{-|t|^p}$, we say that X has stable type p if one can find a constant C such that for every finite family $(x_i)_{i=1}^n$ in X, we have

$$\mathbb{E}\Big(\Big\| \sum_{i=1}^n \xi_i x_i \Big\|^{\frac{p}{2}} \Big)^{2/p} \leq C \Big(\sum_{i=1}^n \|x_i\|^p \Big)^{1/p}.$$

We can now state this important result about types proved by Pisier (obtained by combining results from [91] and [90]).

Theorem 4.1.1. *Let X be a normed space. The following assertions are equivalent:*
(i) X does not contain the ℓ_1^n's uniformly;
(ii) X has infratype p for some $p \in (1, \infty)$;
(iii) X has Rademacher type p for some $p \in (1, 2]$;
(iv) X has stable type $1 + \varepsilon$ for some $\varepsilon \in (0, 1]$;
(v) X has stable type 1.

For more information about the different notions of type and related results, we refer the reader to [88] or [77].

4.2 Preliminaries

4.2.1 Asymptotic B-convexity and asymptotic finite representability

Let us start this subsection by introducing asymptotic B-convexity.

Definition 4.2.1. Let $k \in \mathbb{N}$, $\varepsilon > 0$.
We say that a Banach space X is (k, ε) asymptotically convex if for every weakly null tree $(x_{\overline{n}})_{\overline{n} \in [\mathbb{N}]^{\leq k}} \subset B_X$, there exists $\overline{n} \in [\mathbb{N}]^k$ such that

$$\inf_{(\varepsilon_j)_{j=1}^k \in \{\pm 1\}^k} \left\| \sum_{j=1}^k \varepsilon_j x_{(n_1,\ldots,n_j)} \right\| \leq k(1-\varepsilon).$$

We say that X is asymptotically B-convex if it is (k, ε) asymptotically convex for some $k \in \mathbb{N}$, $\varepsilon > 0$.

In the local theory, if a Banach space X is $(2, \varepsilon)$ convex for some $\varepsilon > 0$, then X is super-reflexive (see [56]). In the asymptotic theory though, being $(2, \varepsilon)$ asymptotically convex does not imply reflexivity because of the following result.

Proposition 4.2.2. *If X is a separable AUS-able Banach space, then X can be renormed in such a way that X is $(2, \varepsilon)$ asymptotically convex for some $\varepsilon > 0$.*

Proof. As said in the proof of Theorem 4.15 of [72], X can be renormed in such a way that there exists $p \in (1, \infty)$ so that

$$\limsup \|x + x_n\|^p \leq \|x\|^p + \limsup \|x_n\|^p$$

whenevery $x \in X$ and $(x_n)_{n \in \mathbb{N}}$ is a weakly null sequence in X (the reader can find a proof of this result in [84]).
Thus it immediately follows that X is $(2, \varepsilon)$ asymptotically convex as soon as $\varepsilon < 1 - 2^{1/p-1}$. □

The next definition was first introduced in [29]. We give it in the general case for $p \in [1, \infty)$ but only $p = 1$ will be of interest in what follows.

Definition 4.2.3. Let $p \in [1, \infty)$. We say that ℓ_p is asymptotically finitely representable in X if for every $\varepsilon > 0$, every $k \in \mathbb{N}$, there exists a weakly null tree $(x_{\overline{n}})_{\overline{n} \in [\mathbb{N}]^{\leq k}} \subset B_X$ such that

$$(1-\varepsilon)\|a\|_{\ell_p^k} \leq \left\| \sum_{j=1}^k a_j x_{(n_1,\ldots,n_j)} \right\| \leq (1+\varepsilon)\|a\|_{\ell_p^k}$$

for all $\overline{n} \in [\mathbb{N}]^k$ and all $a = (a_1, \cdots, a_k) \in \mathbb{R}^k$. We say that ℓ_p is asymptotically crudely finitely representable in X if we can find $C > 0$ such that, for every $k \in \mathbb{N}$, there exists a weakly null tree $(x_{\overline{n}})_{\overline{n} \in [\mathbb{N}]^{\leq k}} \subset B_X$ such that

$$\frac{1}{C}\|a\|_{\ell_p^k} \leq \left\| \sum_{j=1}^k a_j x_{(n_1,\ldots,n_j)} \right\| \leq C\|a\|_{\ell_p^k}$$

for all $\overline{n} \in [\mathbb{N}]^k$ and all $a = (a_1, \cdots, a_k) \in \mathbb{R}^k$.

Let us notice that "ℓ_p is asymptotically crudely finitely representable in X" only means in other words that the ℓ_p^n, $n \in \mathbb{N}$, are uniformly present in the asymptotic structure of X but naming it that way makes it easier when it comes to make parallels with the local results.

4.2.2 Asymptotic types

Let us finish this subsection by introducing the asymptotic versions of the types involved in Theorem 4.0.1.

First, we recall the reader that a random variable ξ on a probability space is called p-stable ($0 < p \leq 2$) if its characteristic function is of the form $t \in \mathbb{R} \mapsto \mathbb{E}(e^{it\xi}) = e^{-c|t|^p}$, for some positive constant $c = c(p)$. It is known that, if ξ is a p-stable random variable, then $\xi \notin L^p$ and $\xi \in L^q$ for all $0 < q < p$ (see [2] for example). In this chapter, we will always assume that $c = 1$ when considering stable variables.

We also want to recall two famous inequalities.

Theorem 4.2.4 (Kahane Theorem [62]). *Let X be a Banach space and let (ε_n) be a sequence of i.i.d Rademacher random variables.*
Then, for all $p, q > 0$, there exists a constant $K > 0$ such that, for any finite sequence $(x_n) \subset X$,

$$\mathbb{E}\left(\Big\|\sum_n \varepsilon x_n\Big\|^q\right)^{1/q} \leq K\mathbb{E}\left(\Big\|\sum_n \varepsilon x_n\Big\|^p\right)^{1/p}.$$

Theorem 4.2.5 (Hoffmann-Jørgensen Theorem [53]). *Let X be a Banach space and let (ξ_n) be a sequence of i.i.d s-stable random variables, $0 < s \leq 2$.*
Then, for each $p, q \in (0, s)$ if $s < 2$, and each $p, q \in (0, \infty)$ if $s = 2$, there exists a constant $K > 0$ such that, for any finite sequence $(x_n) \subset X$,

$$\mathbb{E}\left(\Big\|\sum_n \xi_n x_n\Big\|^q\right)^{1/q} \leq K\mathbb{E}\left(\Big\|\sum_n \xi_n x_n\Big\|^p\right)^{1/p}.$$

Let X be a Banach space. We now introduce the asymptotic types.

Definition 4.2.6. • Let $p \in [1, \infty]$. We say that X has asymptotic infratype p if there exists $C > 0$ such that for each $k \in \mathbb{N}$, every weakly null tree $(x_{\overline{n}})_{\overline{n} \in [\mathbb{N}]^{\leq k}} \subset B_X$, every $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$, we can find $\overline{n} \in [\mathbb{N}]^k$ such that

$$\min_{(\varepsilon_j)_{j=1}^k \in \{\pm 1\}^k} \Big\|\sum_{j=1}^k \varepsilon_j a_j x_{(n_1,\ldots,n_j)}\Big\| \leq C\|a\|_{\ell_p^k}.$$

• Let $p \in [1, \infty)$ and (ε_n) be a sequence of i.i.d Rademacher random variables.
We say that a Banach space X has asymptotic Rademacher type p if there exists $C > 0$ such that, for each $k \in \mathbb{N}$, for every weakly null tree $(x_{\overline{n}})_{\overline{n} \in [\mathbb{N}]^{\leq k}} \subset B_X$, for every $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$, we can find $\overline{n} \in [\mathbb{N}]^k$ such that

$$\mathbb{E}\left(\Big\|\sum_{j=1}^k \varepsilon_j a_j x_{(n_1,\ldots,n_j)}\Big\|^p\right)^{1/p} \leq C\|a\|_{\ell_p^k}.$$

• Let $p \in (0,2]$ and (ξ_i) be a sequence of i.i.d p-stable random variables with the characteristic function $t \in \mathbb{R} \mapsto \mathbb{E}(e^{it\xi_1}) = e^{-|t|^p}$.
We say that a Banach space X has asymptotic stable type p if there exists $C > 0$ such that, for each $k \in \mathbb{N}$, for every weakly null tree $(x_{\overline{n}})_{\overline{n} \in [\mathbb{N}]^{\leq k}} \subset B_X$, for every $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$, we can find $\overline{n} \in [\mathbb{N}]^k$ such that

$$\mathbb{E}\left(\Big\| \sum_{j=1}^{k} \xi_j a_j x_{(n_1,\ldots,n_j)} \Big\|^{\frac{p}{2}}\right)^{2/p} \leq C\|a\|_{\ell_p^k}.$$

By Kahane Theorem and Hoffmann-Jørgensen Theorem, we can immediately note the following.

Remark 4.2.7. • By Kahane Theorem, X has asymptotic Rademacher type p iff for every $q \in [1,\infty)$, there exists $C > 0$ such that, for each $k \in \mathbb{N}$, for every weakly null tree $(x_{\overline{n}})_{\overline{n} \in [\mathbb{N}]^{\leq k}} \subset B_X$, for every $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$, we can find $\overline{n} \in [\mathbb{N}]^k$ such that

$$\mathbb{E}\left(\Big\| \sum_{j=1}^{k} \varepsilon_j a_j x_{(n_1,\ldots,n_j)} \Big\|^{q}\right)^{1/q} \leq C\|a\|_{\ell_p^k}.$$

• By Hoffmann-Jørgensen Theorem, X has asymptotic stable type $p \in (0,2)$ iff for every $q \in (0,p)$, there exists $C > 0$ such that, for all $k \in \mathbb{N}$, for every weakly null tree $(x_{\overline{n}})_{\overline{n} \in [\mathbb{N}]^{\leq k}} \subset B_X$, for all $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$, we can find $\overline{n} \in [\mathbb{N}]^k$ such that

$$\mathbb{E}\left(\Big\| \sum_{j=1}^{k} \xi_j a_j x_{(n_1,\ldots,n_j)} \Big\|^{q}\right)^{1/q} \leq C\|a\|_{\ell_p^k}$$

and X has asymptotic stable type 2 iff for every $q \in (0,\infty)$, there exists $C > 0$ such that, for all $k \in \mathbb{N}$, for every weakly null tree $(x_{\overline{n}})_{\overline{n} \in [\mathbb{N}]^{\leq k}} \subset B_X$, for all $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$, we can find $\overline{n} \in [\mathbb{N}]^k$ such that

$$\mathbb{E}\left(\Big\| \sum_{j=1}^{k} \xi_j a_j x_{(n_1,\ldots,n_j)} \Big\|^{q}\right)^{1/q} \leq C\|a\|_{\ell_2^k}.$$

Ramsey's Theorem even allows us to exchange some quantifiers. We only prove it for asymptotic Rademacher type but a similar result can be proved for the others.

Lemma 4.2.8. *Let $p \in (1,\infty)$. Then X has asymptotic Rademacher type p iff there exists $D > 0$ such that, for all $k \in \mathbb{N}$, for every weakly null tree $(x_{\overline{n}})_{\overline{n} \in [\mathbb{N}]^{\leq k}} \subset B_X$, we can find $\overline{n} \in [\mathbb{N}]^k$ such that, for all $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$, we have:*

$$\mathbb{E}\left(\Big\| \sum_{j=1}^{k} \varepsilon_j a_j x_{(n_1,\ldots,n_j)} \Big\|\right) \leq D\|a\|_{\ell_p^k}.$$

By Kahane Theorem, we immediatly deduce that X has asymptotic Rademacher type p iff, for every $q \in [1,\infty)$, there exists $D > 0$ such that, for all $k \in \mathbb{N}$, for every weakly null tree $(x_{\overline{n}})_{\overline{n} \in [\mathbb{N}]^{\leq k}} \subset B_X$, we can find $\overline{n} \in [\mathbb{N}]^k$ such that, for all $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$, we have:

$$\mathbb{E}\left(\Big\| \sum_{j=1}^{k} \varepsilon_j a_j x_{(n_1,\ldots,n_j)} \Big\|^{q}\right)^{1/q} \leq D\|a\|_{\ell_p^k}.$$

Proof. Assume X has asymptotic Rademacher type p. There exists $C > 0$ such that, for all $k \in \mathbb{N}$, for every weakly null tree $(x_{\overline{n}})_{\overline{n} \in [\mathbb{N}]^{\leq k}} \subset B_X$, for all $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$, we can find $\overline{n} \in [\mathbb{N}]^k$ such that

$$\mathbb{E}\left(\left\| \sum_{j=1}^{k} \varepsilon_j a_j x_{(n_1,\ldots,n_j)} \right\|\right) \leq C \|a\|_{\ell_p^k}.$$

Let $k \in \mathbb{N}$, $(x_{\overline{n}})_{\overline{n} \in [\mathbb{N}]^{\leq k}} \subset B_X$ a weakly null tree.
Let $\eta = \frac{1}{k^{1/q}}$ where q is the conjugate exponent of p, N an η-net of $B_{\ell_p^k}$. By Theorem 2.1.1, there exists $\overline{n} \in [\mathbb{N}]^k$ such that

$$\forall b = (b_1, \ldots, b_k) \in N,\ \mathbb{E}\left(\left\| \sum_{j=1}^{k} \varepsilon_j b_j x_{(n_1,\ldots,n_j)} \right\|\right) \leq C.$$

Let $a = (a_1, \ldots, a_k) \in B_{\ell_p^k}$. There exists $b \in N$ so that $\|a - b\|_{\ell_p^k} \leq \eta$. We deduce

$$\begin{aligned}\mathbb{E}\left(\left\| \sum_{j=1}^{k} \varepsilon_j a_j x_{(n_1,\ldots,n_j)} \right\|\right) &\leq C + \mathbb{E}\left(\left\| \sum_{j=1}^{k} \varepsilon_j (a_j - b_j) x_{(n_1,\ldots,n_j)} \right\|\right) \\ &\leq C + k^{1/q}\eta \leq C + 1.\end{aligned}$$

□

In order to follow Pisier's presentation and make some proofs easier to understand, we chose to prove the two following results separately. Let X be a Banach space with separable dual.

Theorem 4.2.9. *The following assertions are equivalent:*
(i) ℓ_1 is not asymptotically finitely representable in X ;
(ii) ℓ_1 is not asymptotically crudely finitely representable in X ;
(iii) X is asymptotically B-convex.

Theorem 4.2.10. *The following assertions are equivalent:*
(i) ℓ_1 is not asymptotically finitely representable in X ;
(ii) X has asymptotic infratype p for some $p > 1$;
(iii) X has asymptotic Rademacher type p for some $p > 1$;
(iv) X has asymptotic stable type p for some $p > 1$;
(v) X has asymptotic stable type 1.

4.3 Asymptotic B-convexity

This subsection is dedicated to the proof of Theorem 4.2.9. We first start with the easiest implication.

Proposition 4.3.1. *If ℓ_1 is not asymptotically finitely representable in X, then X is asymptotically B-convex.*

Proof. Assume that X is not asymptotically B-convex. Let $k \in \mathbb{N}$, $\varepsilon > 0$. Then there exists a weakly null tree $(x_{\overline{n}})_{\overline{n}\in[\mathbb{N}]^{\leq k}} \subset B_X$ such that:

$$\forall \overline{n} \in [\mathbb{N}]^k, \quad \inf_{(\varepsilon_j)_{j=1}^k \in \{\pm 1\}^k} \Big\| \sum_{j=1}^k \varepsilon_j x_{(n_1,\dots,n_j)} \Big\| \geq k - \varepsilon.$$

Let $a = (a_j)_{j=1}^k \in S_{\ell_1^k}$ and $\overline{n} \in [\mathbb{N}]^k$. For every $1 \leq j \leq k$, we denote $\varepsilon_j = \operatorname{sgn}(a_j)$. Then we have:

$$\begin{aligned} k - \varepsilon \leq \Big\| \sum_{j=1}^k \varepsilon_j x_{(n_1,\dots,n_j)} \Big\| &= \Big\| \sum_{j=1}^k [\varepsilon_j(1 - |a_j|) + a_j] x_{(n_1,\dots,n_j)} \Big\| \\ &\leq \sum_{j=1}^k (1 - |a_j|) + \Big\| \sum_{j=1}^k a_j x_{(n_1,\dots,n_j)} \Big\| \\ &= k - 1 + \Big\| \sum_{j=1}^k a_j x_{(n_1,\dots,n_j)} \Big\|. \end{aligned}$$

The result follows by homogeneity. □

We use now the separability of X^*. Let $(x_n^*) \subset X^*$ be a dense sequence. For every $n \in \mathbb{N}$, and every $\varepsilon > 0$, we let

$$V_{n,\varepsilon} = \{x \in X;\ \forall i \in \{1, \cdots, n\},\ |x_i^*(x)| < \varepsilon\}.$$

For every $k \in \mathbb{N}$, let us denote by $\lambda_k(X)$ the smallest possible constant $\lambda \geq 0$ such that for every weakly null tree $(x_{\overline{n}})_{\overline{n}\in[\mathbb{N}]^{\leq k}} \subset B_X$, there exists $\overline{n} \in [\mathbb{N}]^k$ such that

$$\inf_{(\varepsilon_j)_{j=1}^k \in \{\pm 1\}^k} \Big\| \sum_{j=1}^k \varepsilon_j x_{(n_1,\dots,n_j)} \Big\| \leq \lambda k.$$

If ℓ_1 is asymptotically crudely finitely representable in X, is it easy to see that there exists $C > 0$ such that $\lambda_k(X) \geq C$ for every $k \in \mathbb{N}$. In order to prove that X asymptotically B-convex implies ℓ_1 not asymptotically crudely finitely representable in X, it is then enough to prove that if X is asymptotically B-convex, then $\liminf_{n\to\infty} \lambda_n(X) = 0$. To do so, we use the submultiplicativity method developed by Pisier. It is worth noticing that even though the ideas to prove Theorem 4.2.9 and 4.2.10 are the same as Pisier's, one has to use more technical arguments to get the submultiplicativity results when it comes to asymptotic theory.

Lemma 4.3.2. *For all $k, l \in \mathbb{N}$, we have $\lambda_{kl}(X) \leq \lambda_k(X)\lambda_l(X)$.*

Proof. Let $k, l \in \mathbb{N}$, $(x_{\overline{n}})_{\overline{n}\in[\mathbb{N}]^{\leq kl}} \subset B_X$ be a weakly null tree, $\lambda_k > \lambda_k(X)$, $\lambda_l > \lambda_l(X)$. By a Ramsey argument, we can assume that there exist $\varepsilon_1^1, \cdots, \varepsilon_k^1 \in \{\pm 1\}$ such that

$$\forall \overline{n} \in [\mathbb{N}]^k, \quad \Big\| \sum_{j=1}^k \varepsilon_j^1 x_{(n_1,\dots,n_j)} \Big\| \leq k\lambda_k.$$

Let $m \in \mathbb{N}$. We construct $n_1^{1,m} < \cdots < n_k^{1,m}$ such that $\varepsilon_j^1 x_{(n_1^{1,m},\dots,n_j^{1,m})} \in V_{m,\frac{1}{km}}$ for all $1 \leq j \leq k$. Then we let $X_m = \frac{1}{k\lambda_k} \sum_{j=1}^k \varepsilon_j^1 x_{(n_1^{1,m},\dots,n_j^{1,m})}$, where $\sum_{j=1}^k \varepsilon_j^1 x_{(n_1^{1,m},\dots,n_j^{1,m})} \in V_{m,1/m}$. Let us fix $m_1 \in \mathbb{N}$. Again, by a Ramsey argument, we can assume that there exist $\varepsilon_1^2, \cdots, \varepsilon_k^2 \in \{\pm 1\}$ such that

$$\forall \overline{n} \in [\mathbb{N}]^k, n_1 > n_k^{1,m_1} \implies \left\| \sum_{j=1}^k \varepsilon_j^2 x_{(n_1^{1,m_1},\dots,n_k^{1,m_1},n_1,\dots,n_j)} \right\| \leq k\lambda_k.$$

We define $(X_{m_1,m})_{m>m_1}$ as above.
We can therefore build inductively a weakly null tree $(X_{\overline{m}})_{\overline{m}\in[\mathbb{N}]^{\leq l}} \subset B_X$. Now, by definition of $\lambda_l(X)$, we can find $\overline{m} \in [\mathbb{N}]^l$ and $\delta_1, \cdots, \delta_l \in \{\pm 1\}$ such that

$$\left\| \sum_{j=1}^l \delta_j X_{m_1,\dots,m_j} \right\| \leq l\lambda_l.$$

This means that

$$\left\| \sum_{i=1}^l \sum_{j=(i-1)k+1}^{ik} \delta_i \varepsilon_j^i x_{(n_1^{1,m_1},\dots,n_k^{1,m_1},\dots,n_1^{i-1,m_{i-1}},\dots,n_k^{i-1,m_{i-1}},n_1^{i,m_i},\dots,n_j^{i,m_i})} \right\| \leq lk\lambda_k\lambda_k$$

The result follows. $\square$

Let us complete the proof of Theorem 4.2.9. The fact that (ii) implies (i) is clear and we have seen that (i) implies (ii) at the very beginning of this section. Now, if X is asymptotically B-convex, *i.e* $\lambda_k(X) < 1$ for some $k \in \mathbb{N}$, then $\lim_{n\to+\infty} \lambda_{k^n}(X) = 0$ and thus we cannot find $C > 0$ such that $\lambda_l(X) \geq C$ for every $l \in \mathbb{N}$. Therefore, if X is asymptotically B-convex, then ℓ_1 is not asymptotically crudely finitely representable in X.

4.4 Asymptotic analogues of results concerning types

Before proving our main results, we recall the two following results of Pisier that can be found in [90].

Proposition 4.4.1 (Proposition 1 [90]). *Let (φ_n) be a symmetric sequence of integrable real random variables and (x_n) be a finite sequence in X. We have*

$$\forall p \in [1,\infty), \inf_n \|\varphi_n\|_{L_1} \mathbb{E}\left(\left\| \sum \varepsilon_n x_n \right\|^p\right)^{1/p} \leq \mathbb{E}\left(\left\| \sum \varphi_n x_n \right\|^p\right)^{1/p}.$$

Lemma 4.4.2 (Lemme 2 [90]). *Let $a = (a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers, let $0 < r < q < p \leq 2$, let (ξ_n) be a sequence of i.i.d q-stable random variables. We have*

$$\|\xi_1\|_r \|a\|_{\ell_q} \leq \mathbb{E}\left(\left(\sum |a_n|^p |\xi_n|^p\right)^{r/p}\right)^{1/r} \leq \frac{\|\xi_1\|_r \|g_1\|_q}{\|g_1\|_r} \|a\|_{\ell_q}$$

where g_1 is a p-stable random variable.

Then, we can prove an asymptotic version of Corollaire 1 and Proposition 3 of [90].

Proposition 4.4.3. *Let $p \in [1,2]$.*
(i) If X has asymptotic stable type p, then X has asymptotic Rademacher type p.
(ii) If X has asymptotic Rademacher type p, then X has asymptotic stable type q for all $q \in (0,p)$.

Proof. (i) We can assume $p > 1$. Let $C > 0$ given by the Definition 4.2.6 of asymptotic stable type p.
Let $k \in \mathbb{N}$, $(x_{\overline{n}})_{\overline{n}\in[\mathbb{N}]^{\leq k}} \subset B_X$ be a weakly null tree, $a = (a_1,\dots,a_k) \in \mathbb{R}^k$. We can find $\overline{n} \in [\mathbb{N}]^k$ such that

$$\mathbb{E}\left(\left\|\sum_{j=1}^k \xi_j a_j x_{(n_1,\dots,n_j)}\right\|\right) \leq C\|a\|_{\ell_p^k}.$$

To conclude, we use Remark 4.2.7 and apply Proposition 4.4.1 with $x_j = a_j x_{(n_1,\dots,n_j)}$, $p = 1$ et $\varphi_j = \xi_j$.
(ii) Let $q \in (0,p)$, $r \in (0,q)$, (ξ_n) q-stable random variables. By Lemma 4.2.8, there exists $C > 0$ such that, for all $k \in \mathbb{N}$, for every weakly null tree $(x_{\overline{n}})_{\overline{n}\in[\mathbb{N}]^{\leq k}} \subset B_X$, we can find $\overline{n} \in [\mathbb{N}]^k$ such that, for all $a = (a_1,\dots,a_k) \in \mathbb{R}^k$, we have:

$$\mathbb{E}\left(\left\|\sum_{j=1}^k \varepsilon_j a_j x_{(n_1,\dots,n_j)}\right\|^r\right)^{1/r} \leq C\|a\|_{\ell_p^k}.$$

Let $k \in \mathbb{N}$, $(x_{\overline{n}})_{\overline{n}\in[\mathbb{N}]^{\leq k}} \subset B_X$ be a weakly null tree, $\overline{n} \in [\mathbb{N}]^k$ given above. Let $a = (a_1,\dots,a_k) \in \mathbb{R}^k$.
For all $\theta \in \Omega$, we have

$$\mathbb{E}_\varepsilon\left(\left\|\sum_{j=1}^k \varepsilon_j \xi_j(\theta) a_j x_{(n_1,\dots,n_j)}\right\|^r\right) \leq C^r\left(\sum_{j=1}^k |a_j|^p |\xi_j(\theta)|^p\right)^{r/p}.$$

Therefore

$$\begin{aligned}\mathbb{E}\left(\left\|\sum_{j=1}^k \xi_j a_j x_{(n_1,\dots,n_j)}\right\|^r\right)^{1/r} &= \left[\mathbb{E}_\xi\mathbb{E}_\varepsilon\left(\left\|\sum_{j=1}^k \xi_j\varepsilon_j a_j x_{(n_1,\dots,n_j)}\right\|^r\right)\right]^{1/r} \\ &\leq \mathbb{E}_\xi\left(C^r\left(\sum_{j=1}^k |a_j|^p|\xi_j|^p\right)^{r/p}\right)^{1/r}.\end{aligned}$$

The conclusion follows from Lemma 4.4.2. □

We are now going to prove Theorem 4.2.10. As earlier, the idea is to use Pisier's submultiplicativity method with more technical arguments. We start with a lemma.

Lemma 4.4.4. *Let $p' \in [1,\infty)$, p its conjugate exponent. Assume that $\lambda_l(X) = l^{-1/p'}$ for some $l \in \mathbb{N}$.*
Then X has asymptotic infratype q for all $q < p$.

Proof. Let $q < p$, $k \in \mathbb{N}$, let $(x_{\overline{n}})_{\overline{n}\in[\mathbb{N}]^{\leq k}} \subset B_X$ be a weakly null tree, $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$. For all $r \geq 0$ let us denote

$$A_r = \left\{1 \leq j \leq k; \frac{\|a\|_{\ell_q^k}}{l^{(r+1)/q}} < |a_j| \leq \frac{\|a\|_{\ell_q^k}}{l^{r/q}}\right\} \text{ and } B = \{r \in \mathbb{N}; A_r \neq \varnothing\}.$$

We note that B is finite. Since B is finite, by Ramsey's Theorem, we can find $\overline{n} \in [\mathbb{N}]^k$ such that, for all $r \in B$, we have

$$\min_{(\varepsilon_j)_{j=1}^k \in \{\pm 1\}^k} \Big\| \sum_{j \in A_r} \varepsilon_j a_j x_{(n_1,\ldots,n_j)} \Big\| \leq |A_r| \lambda_{|A_r|}(X) \frac{\|a\|_{\ell_q^k}}{l^{r/q}}.$$

Now, by the triangle inequality, we have

$$\min_{(\varepsilon_j)_{j=1}^k \in \{\pm 1\}^k} \Big\| \sum_{j=1}^k \varepsilon_j a_j x_{(n_1,\ldots,n_j)} \Big\| \leq \sum_{r \in B} \min_{(\varepsilon_j)_{j=1}^k \in \{\pm 1\}^k} \Big\| \sum_{j \in A_r} \varepsilon_j a_j x_{(n_1,\ldots,n_j)} \Big\|$$

and $\|a\|_{\ell_q^k} \geq |A_r|^{1/q} \frac{\|a\|_{\ell_q^k}}{l^{(r+1)/q}}$ so $|A_r| \leq l^{r+1}$. Since the sequence $(n\lambda_n(X))_{n\in\mathbb{N}}$ is clearly non-decreasing, we can deduce that

$$\begin{aligned}
\min_{(\varepsilon_j)_{j=1}^k \in \{\pm 1\}^k} \Big\| \sum_{j=1}^k \varepsilon_j a_j x_{(n_1,\ldots,n_j)} \Big\| &\leq \sum_{r \in B} |A_r| \lambda_{|A_r|}(X) \frac{\|a\|_{\ell_q^k}}{l^{r/q}} \\
&\leq \sum_{r \in B} \frac{l^{r+1} \lambda_{l^{r+1}}(X)}{l^{r/q}} \|a\|_{\ell_q^k} \\
&\leq \sum_{r \in B} \frac{l^{r+1} \lambda_l(X)^{r+1}}{l^{r/q}} \|a\|_{\ell_q^k} \\
&= \sum_{r \in B} \frac{l^{r+1}}{l^{r/q} l^{(r+1)/p'}} \|a\|_{\ell_q^k} \\
&\leq l^{1/p} \left(\sum_{r=0}^{\infty} (l^{1-1/q-1/p'})^r \right) \|a\|_{\ell_q^k}.
\end{aligned}$$

We conclude by noting that $\sum_{r=0}^{\infty} (l^{1-1/q-1/p'})^r < \infty$. □

From this, we can immediately deduce the first part of the following result, which is an asymptotic version of Proposition 2 of [91]. We prove the second part to continue the parallel with the local situation even though we will not use it.

Proposition 4.4.5. *X is asymptotically B-convex if and only if X has asymptotic infratype p for some $p > 1$.*
Moreover, if we denote by $\Lambda(X)$ the supremum of all $p \geq 1$ such that X has infratype p, then

$$\Lambda(X) = \lim_{n \to +\infty} \frac{\ln(n)}{\ln(n\lambda_n(X))}.$$

Proof. By definition, X is asymptotically B-convex iff $\lambda_n(X) < 1$ for some $n \geq 2$, *i.e* iff there exists $p' \in [1, \infty)$ such that $\lambda_n(X) = \frac{1}{n^{1/p'}}$. So the first part of the proposition follows from Lemma 4.4.4.
Let us prove the second part of the proposition.
• Let p such that X has infratype p. There exists $C > 0$ such that for all $n \geq 2$, $n\lambda_n(X) \leq Cn^{1/p}$ so

$$\liminf \frac{\ln(n)}{\ln(n\lambda_n(X))} \geq \liminf \frac{\ln(n)}{\ln(C) + \frac{1}{p}\ln(n)} = p.$$

Thus $\liminf \frac{\ln(n)}{\ln(n\lambda_n(X))} \geq \Lambda(X)$.
• If $q > \Lambda(X)$, by Lemma 4.4.4, we have $n\lambda_n(X) \geq n^{1/q}$ for all $n \geq 2$. Therefore

$$\limsup \frac{\ln(n)}{\ln(n\lambda_n(X))} \leq \limsup \frac{\ln(n)}{\frac{1}{q}\ln(n)} = q.$$

Thus $\limsup \frac{\ln(n)}{\ln(n\lambda_n(X))} \leq \Lambda(X)$. □

To prove Theorem 4.2.10, as said before, we will use another submultiplicativity argument. To do so, we introduce a few notation.
For $k \in \mathbb{N}$, we will denote by $\mu_k(X)$ the smallest possible constant $\mu \geq 0$ such that for every weakly null tree $(x_{\overline{n}})_{\overline{n} \in [\mathbb{N}]^{\leq k}} \subset B_X$, we can find $\overline{n} \in [\mathbb{N}]^k$ so that

$$\mathbb{E}\left(\Big\|\sum_{j=1}^{k} \varepsilon_j x_{(n_1,\ldots,n_j)}\Big\|^2\right)^{1/2} \leq \mu k.$$

Since the map $k \mapsto \mu_k(X)$ is not submultiplicative a priori, we introduce one more quantity which will be. For $k \in \mathbb{N}$, we will denote by $\nu_k(X)$ the smallest possible constant $\nu \geq 0$ such that for every weakly null tree $(x_{\overline{n}})_{\overline{n} \in [\mathbb{N}]^{\leq k}} \subset B_X$, for all $a = (a_1, \cdots, a_k) \in \mathbb{R}^k$, we can find $\overline{n} \in [\mathbb{N}]^k$ so that

$$\mathbb{E}\left(\Big\|\sum_{j=1}^{k} \varepsilon_j a_j x_{(n_1,\ldots,n_j)}\Big\|^2\right)^{1/2} \leq \nu\sqrt{k}\|a\|_{\ell_2^k}.$$

Note that, for all $k \in \mathbb{N}$, we have $\lambda_k(X) \leq \mu_k(X) \leq \nu_k(X) \leq 1$.
The key lemma is the following.

Lemma 4.4.6. *For all $k, l \in \mathbb{N}$, we have $\nu_{kl}(X) \leq \nu_k(X)\nu_l(X)$.*

Proof. Let $k, l \in \mathbb{N}$, $\nu_k > \nu_k(X), \nu_l > \nu_l(X)$, $(x_{\overline{n}})_{\overline{n} \in [\mathbb{N}]^{\leq k}} \subset B_X$ a weakly null tree, $a = (a_1, \cdots, a_{kl}) \in \mathbb{R}^{kl}$.
By a Ramsey argument, we can assume that

$$\forall \overline{n} \in [\mathbb{N}]^k, \mathbb{E}\left(\Big\|\sum_{j=1}^{k} \varepsilon_j a_j x_{(n_1,\ldots,n_k)}\Big\|^2\right)^{1/2} \leq \nu_k\sqrt{k}\left(\sum_{j=1}^{k} |a_j|^2\right)^{1/2}.$$

We recall that for every $n \in \mathbb{N}$, and every $\varepsilon > 0$, we let

$$V_{n,\varepsilon} = \{x \in X;\ \forall i \in \{1, \cdots, n\},\ |x_i^*(x)| < \varepsilon\}.$$

Let $m \in \mathbb{N}$. We construct $n_1^{1,m} < \cdots < n_k^{1,m}$ such that $a_j x_{(n_1^{1,m},\ldots,n_j^{1,m})} \in V_{m,\frac{1}{km}}$ for all $1 \leq j \leq k$. Then, for every $\delta = (\delta_1, \cdots, \delta_k) \in \{\pm 1\}^k$, we let $X_m(\delta) = \sum_{j=1}^k \delta_j a_j x_{(n_1^{1,m},\ldots,n_j^{1,m})} \in V_{m,1/m}$.
Let us fix $m_1 \in \mathbb{N}$. Again, by a Ramsey argument, we can assume that

$$\forall \overline{n} \in [\mathbb{N}]^k, n_1 > n_k^{1,m_1} \implies \mathbb{E}\left(\Big\| \sum_{j=1}^k \varepsilon_j a_{k+j} x_{(n_1^{1,m_1},\ldots,n_k^{1,m_1},n_1,\ldots,n_j)} \Big\|^2\right)^{1/2} \leq \nu_k \sqrt{k} \left(\sum_{j=k+1}^{2k} |a_j|^2\right)^{1/2}.$$

For every $\delta = (\delta_1, \cdots, \delta_k) \in \{\pm 1\}^k$, we define $(X_{m_1,m}(\delta))_{m>m_1}$ as above.
For every $(\delta_1^1, \cdots, \delta_k^1, \delta_1^2, \cdots, \delta_k^2, \cdots, \delta_1^l, \cdots, \delta_k^l) \in \{\pm 1\}^{kl}$, we can therefore build inductively a weakly null tree $(X_{\overline{m}}((\delta_1^{|\overline{m}|}, \cdots, \delta_k^{|\overline{m}|}))_{\overline{m} \in [\mathbb{N}]^{\leq l}}$. Let $(\eta_j)_{j\in\mathbb{N}}$ be a sequence of iid Rademacher variables independant of $(\varepsilon_j)_{j\in\mathbb{N}}$. By definition of $\nu_l(X)$, since $\{\pm 1\}^{kl}$ is finite, we can find $\overline{m} \in [\mathbb{N}]^l$ so that

$$\forall (\delta_1^1, \cdots, \delta_k^1, \cdots, \delta_1^l, \cdots, \delta_k^l) \in \{\pm 1\}^{kl},\ \mathbb{E}_\eta\Big(\Big\| \sum_{j=1}^l \eta_j X_{(m_1,\ldots,m_j)}(\delta^j) \Big\|^2\Big) \leq \nu_l^2 l \Big(\sum_{j=1}^l \|X_{(m_1,\ldots,m_j)}(\delta^j)\|^2\Big).$$

This ensures that

$$\frac{1}{2^{kl}} \sum_{\substack{(\delta_i^j)_{1\leq i\leq k} \in \{\pm 1\}^{kl} \\ 1 \leq j \leq l}} \mathbb{E}_\eta\Big(\Big\| \sum_{i=1}^l \sum_{j=(i-1)k+1}^{ik} \eta_i \delta_{j-(i-1)k}^i a_j x_{(n_1^{1,m_1},\ldots,n_k^{1,m_1},\ldots,n_1^{i-1,m_{i-1}},\ldots,n_k^{i-1,m_{i-1}},n_1^{i,m_i},\ldots,n_j^{i,m_i})} \Big\|^2\Big)$$

$$\leq \frac{1}{2^{kl}} \sum_{\substack{(\delta_i^j)_{1\leq i\leq k} \in \{\pm 1\}^{kl} \\ 1 \leq j \leq l}} \nu_l^2 l \Big(\sum_{i=1}^l \Big\| \sum_{j=(i-1)k+1}^{ik} \delta_{j-(i-1)k}^i a_j x_{(n_1^{1,m_1},\ldots,n_k^{1,m_1},\ldots,n_1^{i-1,m_{i-1}},\ldots,n_k^{i-1,m_{i-1}},n_1^{i,m_i},\ldots,n_j^{i,m_i})} \Big\|^2\Big)$$

$$\leq \nu_l^2 l \times \nu_k^2 k \sum_{i=1}^l \sum_{j=(i-1)k+1}^{ik} |a_j|^2 = kl \nu_k^2 \nu_l^2 \|a\|_{\ell_2^{kl}}^2.$$

Since

$$\frac{1}{2^{kl}} \sum_{\substack{(\delta_i^j)_{1\leq i\leq k} \in \{\pm 1\}^{kl} \\ 1 \leq j \leq l}} \mathbb{E}_\eta\Big(\Big\| \sum_{i=1}^l \sum_{j=(i-1)k+1}^{ik} \eta_i \delta_{j-(i-1)k}^i a_j x_{(n_1^{1,m_1},\ldots,n_k^{1,m_1},\ldots,n_1^{i-1,m_{i-1}},\ldots,n_k^{i-1,m_{i-1}},n_1^{i,m_i},\ldots,n_j^{i,m_i})} \Big\|^2\Big)$$

$$= \mathbb{E}\left(\Big\| \sum_{i=1}^l \sum_{j=(i-1)k+1}^{ik} \varepsilon_j a_j x_{(n_1^{1,m_1},\ldots,n_k^{1,m_1},\ldots,n_1^{i-1,m_{i-1}},\ldots,n_k^{i-1,m_{i-1}},n_1^{i,m_i},\ldots,n_j^{i,m_i})} \Big\|^2\right),$$

we get our result. □

As before, we get an asymptotic version of Lemme 4 of [91].

Lemma 4.4.7. *Let $p' \in [1, \infty)$, p its conjugate exponent. Assume that $\nu_l(X) = l^{-1/p'}$ for some $l \in \mathbb{N}$.*
Then X is of asymptotic Rademacher type q for all $q < p$.

Proof. To prove this result, we just copy the proof of Lemma 4.4.4, using the fact that $\mu_k(X) \leq \nu_k(X)$ for all $k \in \mathbb{N}$, and Remark 4.2.7. □

Now, the relation we need between the quantities $\lambda_k(X)$, $\mu_k(X)$ and $\nu_k(X)$ is given in the following lemma.

Lemma 4.4.8. *For every $k \in \mathbb{N}$, we have*

$$\lambda_k(X) = 1 \iff \mu_k(X) = 1 \iff \nu_k(X) = 1.$$

Proof. $*$ Let $k \in \mathbb{N}$. Let us prove that $\mu_k(X) = 1 \implies \lambda_k(X) = 1$.
Let $\mu < \mu_k(X)$. There exists a weakly null tree $(x_{\overline{n}})_{\overline{n} \in [\mathbb{N}]^{\leq k}} \subset B_X$ such that

$$\forall \overline{n} \in [\mathbb{N}]^k, \ \mathbb{E}\left(\left\| \sum_{j=1}^{k} \varepsilon_j x_{(n_1,\dots,n_j)} \right\|^2\right)^{1/2} \geq \mu k$$

and there exists $\overline{n} \in [\mathbb{N}]^k$ such that $\inf_{(\varepsilon_j)_{j=1}^k \in \{\pm 1\}^k} \left\| \sum_{j=1}^k \varepsilon_j x_{(n_1,\dots,n_j)} \right\| \leq \lambda_k(X) k.$

Let us first notice that

$$\mathbb{E}\left(\left\| \sum_{j=1}^{k} \varepsilon_j y_j \right\|^2\right) \leq \frac{\inf_{(\varepsilon_j)_{j=1}^k \in \{\pm 1\}^k} \left\| \sum_{j=1}^k \varepsilon_j y_j \right\|^2 + (2^{k-1} - 1)k^2}{2^{k-1}}$$

for all $y_1, \cdots, y_k \in X$. Indeed, we can find $(\tilde{\varepsilon}_j)_{j=1}^k \in \{\pm 1\}^k$ such that

$$\inf_{(\varepsilon_j)_{j=1}^k \in \{\pm 1\}^k} \left\| \sum_{j=1}^{k} \varepsilon_j y_j \right\|^2 = \left\| \sum_{j=1}^{k} \tilde{\varepsilon}_j y_j \right\|^2$$

so

$$\begin{aligned}
\mathbb{E}\left(\left\| \sum_{j=1}^{k} \varepsilon_j y_j \right\|^2\right) &= \frac{1}{2^k}\Big(2 \inf_{(\varepsilon_j)_{j=1}^k \in \{\pm 1\}^k} \left\| \sum_{j=1}^{k} \varepsilon_j y_j \right\|^2 + \sum_{(\varepsilon_j)_{j=1}^k \in \{\pm 1\}^k \setminus \{(\tilde{\varepsilon}_j)_{j=1}^k, (-\tilde{\varepsilon}_j)_{j=1}^k\}} \left\| \sum_{j=1}^{k} \varepsilon_j y_j \right\|^2\Big) \\
&\leq \frac{1}{2^k}\Big(2 \inf_{(\varepsilon_j)_{j=1}^k \in \{\pm 1\}^k} \left\| \sum_{j=1}^{k} \varepsilon_j y_j \right\|^2 + (2^k - 2)k^2\Big).
\end{aligned}$$

Thus, with $y_j = x_{(n_1,\dots,n_j)}$ for $1 \leq j \leq k$, we have

$$\begin{aligned}
\mu k &\leq \mathbb{E}\left(\left\| \sum_{j=1}^{k} \varepsilon_j x_{(n_1,\dots,n_j)} \right\|^2\right)^{1/2} \\
&\leq \left(\frac{\inf_{(\varepsilon_j)_{j=1}^k \in \{\pm 1\}^k} \left\| \sum_{j=1}^k \varepsilon_j x_{(n_1,\dots,n_j)} \right\|^2 + (2^{k-1} - 1)k^2}{2^{k-1}}\right)^{1/2} \\
&\leq k \left(\frac{\lambda_k(X)^2 + 2^{k-1} - 1}{2^{k-1}}\right)^{1/2}
\end{aligned}$$

and so $2^{k-1}\mu^2 \leq \lambda_k(X)^2 + 2^{k-1} - 1$.
Then, for all $\mu < \mu_k(X)$, we have $2^{k-1}(\mu^2 - 1) \leq \lambda_k(X)^2 - 1$. As a consequence, we have the desired implication.
$*$ Assume now that $\nu_k(X) = 1$ and let us prove that $\mu_k(X) = 1$.
Let $\varepsilon > 0$. We can find a weakly null tree $(x_{\overline{n}})_{\overline{n}\in[\mathbb{N}]^{\leq k}} \subset B_X$ and $a = (a_1, \cdots, a_k) \in \mathbb{R}^k$ such that $\|a\|_{\ell_2^k} = \sqrt{k}$ and

$$\forall \overline{n} \in [\mathbb{N}]^k,\ \mathbb{E}\left(\left\|\sum_{j=1}^k \varepsilon_j a_j x_{(n_1,\ldots,n_j)}\right\|^2\right)^{1/2} \geq (1-\varepsilon)\sqrt{k}\|a\|_{\ell_2^k} = (1-\varepsilon)k.$$

The computation in [91] (in the proof of Lemme 5) gives us $\max\limits_{1\leq i\leq n} |a_i| \leq 1 + 2\sqrt{k\varepsilon}$.
Now, if we put $y_{(n_1,\ldots,n_j)} = \frac{a_j}{1+2\sqrt{k\varepsilon}} x_{(n_1,\ldots,n_j)}$ for every $(n_1,\ldots,n_j) \in [\mathbb{N}]^{\leq k}$, the tree $(y_{\overline{n}})_{\overline{n}\in[\mathbb{N}]^{\leq k}} \subset B_X$ is weakly null and

$$\forall \overline{n} \in [\mathbb{N}]^k, \mathbb{E}\left(\left\|\sum_{j=1}^k \varepsilon_j y_{(n_1,\ldots,n_j)}\right\|^2\right)^{1/2} \geq \frac{1-\varepsilon}{1+2\sqrt{k\varepsilon}}k.$$

This is true for every $\varepsilon > 0$ so we get our implication.
$*$ We conclude the proof by remembering that $\lambda_k(X) \leq \mu_k(X) \leq \nu_k(X) \leq 1$. □

We are now able to prove Theorem 4.2.10.

Proof of Theorem 4.2.10. $(i) \implies (ii)$ was already proved.
$(ii) \implies (iii)$ By assumption, there exists $k \in \mathbb{N}$ such that $\lambda_k(X) < 1$ and so, by Lemma 4.4.8, $\nu_k(X) < 1$. Lemma 4.4.7 ensures the conclusion.
$(iii) \implies (iv) \implies (v)$ was proved earlier.
$(v) \implies (i)$ Assume that ℓ_1 is asymptotically finitely representable in X and that X has asymptotic stable type 1 with constant C. Let $a = (a_n) \in \ell_1$. In order to get a contradiction, let us prove that $\sum_n |a_n\xi_n|$ converges a.s.
Let $k \in \mathbb{N}$. By our assumption, there exists a weakly null tree $(x_{\overline{n}})_{\overline{n}\in[\mathbb{N}]^{\leq k}} \subset B_X$ such that

$$\frac{1}{2}\|b\|_{\ell_1^k} \leq \left\|\sum_{j=1}^k b_j x_{n_1,\cdots n_j}\right\| \leq \|b\|_{\ell_1^k}$$

for all $\overline{n} \in [\mathbb{N}]^k$ and all $b = (b_1,\ldots,b_k) \in \mathbb{R}^k$.
Since X has asymptotic stable type 1, there exists $\overline{n} \in [\mathbb{N}]^k$ such that

$$\mathbb{E}\left(\left\|\sum_{j=1}^k \xi_j a_j x_{(n_1,\ldots,n_j)}\right\|^{1/2}\right)^2 \leq C\|(a_1,\ldots,a_k)\|_{\ell_1^k}.$$

We get:

$$\mathbb{E}\left(\left(\sum_{j=1}^k |\xi_j a_j|\right)^{\frac{1}{2}}\right)^2 \leq 2C\|a\|_{\ell_1}.$$

Therefore, $\sum_n |a_n\xi_n|$ converges a.s. Now, since $\sum_n |a_n\xi_n|$ converges a.s if and only if

$$(*) \quad \sum_n |a_n| \left(1 + \ln\left(\frac{1}{|a_n|}\right)\right) < \infty$$

(see [91] and references therein), we get a contraction by considering $a = (a_n) \in \ell_1$ that does not satisfy $(*)$. □

Remark 4.4.9. In [29], it is proved that "ℓ_1 is not asymptotically finitely representable in ·" is a 3SP. Moreover, we get that asymptotic B-convexity is stable under linear isomorphism.

We conclude this subsection with a final remark. In the local theory, X is B-convex if and only if its bidual X^{**} is B-convex. In the asymptotic setting, if we consider the dual of the Lindenstrauss space Z_{c_0}, then $X = Z_{c_0}^*$ is 2-AUS-able so it is asymptotically B-convex, but $X^{**} = X \oplus \ell_1$ contains ℓ_1.
Besides, there is no chance of proving a result of stability under non-linear embedding by Aharoni's Theorem, since c_0 is asymptotically B-convex.

4.5 p-uniform smoothness implies Enflo type p

It is well known that a p-uniformly smooth space has Enflo type p, and we are aware of two proofs. The first one uses Ball's notion of Markov type. It was first shown in [82] that p-uniform smoothness implies Markov type p, and later proved in [83] that Markov type p implies Enflo type p (by essentially looking at the regular random walk on the Hamming cubes). Since linear type p is a straightforward consequence of p-uniform smoothness, the second proof goes via the recent solution of Enflo's problem by Ivanishvilli, van Handel, and Volberg (they showed that Rademacher type p implies Enflo type p). We give here an elementary proof inspired by a martingale approach to Kalton and Randrianarivony's result from [67]. It is likely that our alternate proof of the local result is known to some experts but we could not locate it in the literature (the closest similar argument is possibly the proof of Theorem 7 in [68]). It is quite surprising actually that the first proof of the fact that L_q has Enflo type 2 when $q \geq 2$, is a consequence of a much deeper result from [83] which solves Enflo's Problem for the class of UMD spaces. The argument we give here, combined with the classical smoothness characteristics of L_p, provides a rather elementary proof of the fact that L_p has Enflo type $\min\{p, 2\}$.

Let $p \in [1, 2]$. We recall that a Banach space X is p-uniformly smooth if and only if there exists a constant $C \geq 1$ such that

$$\frac{\|x + y\|^p + \|x - y\|^p}{2} \leq \|x\|^p + C\|y\|^p$$

for all $x, y \in X$ (cf [13]), that X has Enflo type p if there exists a constant $C > 0$ such that, for every $n \in \mathbb{N}$, for every map $f : \{\pm 1\}^n \to X$, we have

$$\mathbb{E}_{\varepsilon\in\{\pm1\}^n}\|f(\varepsilon) - f(-\varepsilon)\|^p \leq C^p \sum_{j=1}^{n} \mathbb{E}_{\varepsilon\in\{\pm1\}^n}\|f(\varepsilon) - f(\varepsilon_1, \dots, \varepsilon_{j-1}, -\varepsilon_j, \varepsilon_{j+1}, \dots, \varepsilon_n)\|^p$$

and we say that X has martingale type p ([101] quoting [92]) if there exists a constant S such that

$$\Big\|\sum_{k=1}^{n} d_k\Big\|_{L_p}^p \leq S^p \sum_{k=1}^{n} \|d_k\|_{L_p}^p$$

for all sequences of X-valued differences $d_1, \cdots, d_n$ of dyadic martingales.

It is known that a Banach space is p-uniformly smooth up to renorming if and only if it has martingale type p (see for instance [93]).

Lemma 4.5.1. *Let Y be a Banach space that has martingale type p. Then there exists $S > 0$ such that, for all $k \in \mathbb{N}$ and all $f : \{\pm 1\}^k \to Y$, we have*

$$\mathbb{E}_{\varepsilon\in\{\pm1\}^k}\|f(\varepsilon) - \mathbb{E}(f)\|^p \leq S^p \sum_{i=1}^{k} \mathbb{E}_{\varepsilon\in\{\pm1\}^i}\mathbb{E}_{(\eta,\delta)\in\{\pm1\}^{k-i+1}}\|f(\varepsilon_1, \cdots, \varepsilon_i, \delta_1, \cdots, \delta_{k-i}) - f(\varepsilon_1, \cdots, \varepsilon_{i-1}, \eta, \delta_1, \cdots, \delta_{k-i})\|^p.$$

Proof. By assumption, there exists $S > 0$ such that Y has martingale type p with constant S. Let $k \in \mathbb{N}$ and $f : \{\pm 1\}^k \to Y$. We define recursively the sequence of Paley-Walsh martingale approximations of f, $(M_i(f) : \{\pm 1\}^k \to Y)_{i=0}^k$, as follows:

1. $M_k(f) = f$;

2. $M_i(f) = \dfrac{M_{i+1}(f)(\varepsilon_1, \cdots, \varepsilon_i, 1) - M_{i+1}(f)(\varepsilon_1, \cdots, \varepsilon_i, -1)}{2}$ for $i \in \{0, \cdots, k-1\}$.

Observe that $M_i(f)(\varepsilon) = \mathbb{E}_{\delta \in \{\pm 1\}^{k-i}} f(\varepsilon_1, \cdots, \varepsilon_i, \delta)$ for every $i \in \{1, \cdots, k\}$ and every $\varepsilon \in \{\pm 1\}^k$, and that $M_0(f) = \mathbb{E}_\varepsilon(f(\varepsilon))$. Since Y has martingale type p, we get

$$\mathbb{E}\|f - M_0(f)\|^p \leq S^p \sum_{i=1}^{k} \mathbb{E}\|M_i(f) - M_{i-1}(f)\|^p.$$

It remains to observe that

$$\begin{aligned} \mathbb{E}\|M_i(f) - M_{i-1}(f)\|^p &= \mathbb{E}_{\varepsilon \in \{\pm 1\}^i} \|\mathbb{E}_{(\eta, \delta) \in \{\pm 1\}^{k-i+1}} f(\varepsilon_1, \cdots, \varepsilon_i, \delta_1, \cdots, \delta_{k-i}) - \\ &\quad f(\varepsilon_1, \cdots, \varepsilon_{i-1}, \eta, \delta_1, \cdots, \delta_{k-i})\|^p \\ &\leq \mathbb{E}_{\varepsilon \in \{\pm 1\}^i} \mathbb{E}_{(\eta, \delta) \in \{\pm 1\}^{k-i+1}} \|f(\varepsilon_1, \cdots, \varepsilon_i, \delta_1, \cdots, \delta_{k-i}) - \\ &\quad f(\varepsilon_1, \cdots, \varepsilon_{i-1}, \eta, \delta_1, \cdots, \delta_{k-i})\|^p, \end{aligned}$$

where the last line follows by convexity. □

Definition 4.5.2. For (X, d_X) and (Y, d_Y) two metric spaces, the Y-distorsion of X, denoted by $c_Y(X)$, is the least constant $D \geq 1$ such that there exist $s > 0$ and a map $f : X \to Y$ satisfying

$$\forall x, y \in X, \;\; s \cdot d_X(x, y) \leq d_Y(f(x), f(y)) \leq sD \cdot d_X(x, y).$$

This lemma allows us to prove the following.

Theorem 4.5.3. *Every p-uniformly smooth Banach space Y for some equivalent norm has Enflo type p. In particular, $c_Y(H_k) \gtrsim k^{1-1/p}$, where H_k denotes the Hamming cube of height k: $H_k = \{\pm 1\}^k$ endowed with the Hamming metric d_k ($d_k(\varepsilon, \eta) = \sum_{j; \varepsilon_j \neq \eta_j} 1$ for all $\varepsilon, \eta \in H_k$).*

Proof. First, it follows easily from the triangle inequality and convexity that

$$\mathbb{E}_{\varepsilon \in \{\pm 1\}^k} \|f(\varepsilon) - f(-\varepsilon)\|^p \leq 2^p \mathbb{E}\|f - \mathbb{E}(f)\|^p = 2^p \mathbb{E}\|f - M_0(f)\|^p.$$

Moreover, since Y has martingale type p (see [92] and [89]), the previous lemma ensures the existence of an $S > 0$ such that, for all $k \in \mathbb{N}$ and all $f : \{\pm 1\}^k \to Y$, we have

$$\mathbb{E}_{\varepsilon \in \{\pm 1\}^k} \|f(\varepsilon) - \mathbb{E}(f)\|^p \leq S^p \sum_{i=1}^{k} \mathbb{E}_{\varepsilon \in \{\pm 1\}^i} \mathbb{E}_{(\eta, \delta) \in \{\pm 1\}^{k-i+1}} \|f(\varepsilon_1, \cdots, \varepsilon_i, \delta_1, \cdots, \delta_{k-i}) - f(\varepsilon_1, \cdots, \varepsilon_{i-1}, \eta, \delta_1, \cdots, \delta_{k-i})\|^p.$$

Now,

$$\begin{aligned}
& \mathbb{E}_{\varepsilon\in\{\pm1\}^i}\mathbb{E}_{\delta\in\{\pm1\}^{k-i+1}}\|f(\varepsilon_1,\cdots,\varepsilon_i,\delta_{i+1},\cdots,\delta_k)-f(\varepsilon_1,\cdots,\varepsilon_{i-1},\delta_i,\delta_{i+1},\cdots,\delta_k)\|^p \\
= & \frac{1}{2^{k+1}}\sum_{\varepsilon\in\{\pm1\}^i}\sum_{\delta\in\{\pm1\}^{k-i}}[\|f(\varepsilon_1,\cdots,\varepsilon_i,\delta)-f(\varepsilon_1,\cdots,\varepsilon_{i-1},-1,\delta)\|^p \\
& +\|f(\varepsilon_1,\cdots,\varepsilon_i,\delta)-f(\varepsilon_1,\cdots,\varepsilon_{i-1},1,\delta)\|^p \\
= & \frac{1}{2^{k+1}}\sum_{\varepsilon\in\{\pm1\}^{i-1}}\sum_{\delta\in\{\pm1\}^{k-i}}[\|f(\varepsilon_1,\cdots,\varepsilon_{i-1},-1,\delta)-f(\varepsilon_1,\cdots,\varepsilon_{i-1},-1,\delta)\|^p \\
& +\|f(\varepsilon_1,\cdots,\varepsilon_{i-1},-1,\delta)-f(\varepsilon_1,\cdots,\varepsilon_{i-1},1,\delta)\|^p+\|f(\varepsilon_1,\cdots,\varepsilon_{i-1},1,\delta)-f(\varepsilon_1,\cdots,\varepsilon_{i-1},1,\delta)\|^p \\
& +\|f(\varepsilon_1,\cdots,\varepsilon_{i-1},1,\delta)-f(\varepsilon_1,\cdots,\varepsilon_{i-1},-1,\delta)\|^p] \\
= & \frac{1}{2^{k+1}}\sum_{\varepsilon\in\{\pm1\}^{i-1}}\sum_{\delta\in\{\pm1\}^{k-i}}[\|f(\varepsilon_1,\cdots,\varepsilon_{i-1},-1,\delta)-f(\varepsilon_1,\cdots,\varepsilon_{i-1},1,\delta)\|^p \\
& +\|f(\varepsilon_1,\cdots,\varepsilon_{i-1},1,\delta)-f(\varepsilon_1,\cdots,\varepsilon_{i-1},-1,\delta)\|^p] \\
= & \frac{1}{2^{k+1}}\sum_{\varepsilon\in\{\pm1\}^i}\sum_{\delta\in\{\pm1\}^{k-i}}\|f(\varepsilon_1,\cdots,\varepsilon_i,\delta)-f(\varepsilon_1,\cdots,\varepsilon_{i-1},-\varepsilon_i,\delta)\|^p \\
= & \frac{1}{2^{k+1}}\sum_{\varepsilon\in\{\pm1\}^k}\|f(\varepsilon)-f(\varepsilon_1,\cdots,\varepsilon_{i-1},-\varepsilon_i,\varepsilon_{i+1},\cdots,\varepsilon_k)\|^p \\
= & \frac{1}{2}\mathbb{E}_{\varepsilon\in\{\pm1\}^k}\|f(\varepsilon)-f(\varepsilon_1,\cdots,\varepsilon_{i-1},-\varepsilon_i,\varepsilon_{i+1},\cdots,\varepsilon_k)\|^p.
\end{aligned}$$

Therefore,

$$\mathbb{E}_{\varepsilon\in\{\pm1\}^k}\|f(\varepsilon)-f(-\varepsilon)\|^p\le 2^{p-1}S^p\sum_{i=1}^k\mathbb{E}_{\varepsilon\in\{\pm1\}^k}\|f(\varepsilon)-f(\varepsilon_1,\cdots,\varepsilon_{i-1},-\varepsilon_i,\varepsilon_{i+1},\cdots,\varepsilon_k)\|^p.$$

□

Remark 4.5.4. Note that we also have

$$\mathbb{E}_{\varepsilon,\varepsilon'\in\{\pm1\}^k}\|f(\varepsilon)-f(\varepsilon')\|^p\approx\mathbb{E}\|f-\mathbb{E}(f)\|^p.$$

Using the fact that there is a good proportion of pairs of points that are at distance at least $k/2$ (in fact $\frac{1}{2}\sum_{i=k/2}^k 2^k\binom{k}{i}$ such pairs) and since $\sum_{i=k/2}^k\binom{k}{i}\gtrsim 2^k$, the distortion lower bound also follows from the alternate inequality

$$\begin{aligned}
\mathbb{E}_{\varepsilon,\varepsilon'\in\{\pm1\}^k}\|f(\varepsilon)-f(\varepsilon')\|^p\le S^p\sum_{i=1}^k\mathbb{E}_{\varepsilon\in\{\pm1\}^i}\mathbb{E}_{(\eta,\delta)\in\{\pm1\}^{k-i+1}}\|f(\varepsilon_1,\cdots,\varepsilon_i,\delta_1,\cdots,\delta_{k-i})- \\
f(\varepsilon_1,\cdots,\varepsilon_{i-1},\eta,\delta_1,\cdots,\delta_{k-i})\|^p
\end{aligned}$$

and the fact that

$$\forall\varepsilon\in\{\pm1\}^k\ \|f(\varepsilon)-f(\varepsilon_1,\cdots,\varepsilon_{i-1},-\varepsilon_i,\varepsilon_{i+1},\cdots,\varepsilon_k)\|^p\le\mathrm{Lip}(f)^p.$$

4.6 Final remarks and open problems

4.6.1 A remark concerning concentration properties of Lipschitz maps defined on the Hamming graphs

For a reflexive Banach space X, the weakest linear condition one can think of in order to have a concentration property that prevents the equi-Lipschitz embedding of the Hamming graphs would be the existence of a function $\varphi : \mathbb{N} \to \mathbb{R}^+$ satisfying $\lim_{k\to\infty} \frac{\varphi(k)}{k} = 0$ and, for every $k \in \mathbb{N}$, every weakly null tree $(x_{\overline{n}})_{\overline{n}\in[\mathbb{N}]^{\leq k}} \subset B_X$, there exists $\overline{n} \in [\mathbb{N}]^k$ so that

$$\Big\| \sum_{j=1}^{k} x_{(n_1,\dots,n_j)} \Big\| \leq \varphi(k).$$

However, if we denote $\gamma_k(X)$, for $k \in \mathbb{N}$, the smallest possible constant γ such that for every weakly null tree $(x_{\overline{n}})_{\overline{n}\in[\mathbb{N}]^{\leq k}} \subset B_X$, there exists $\overline{n} \in [\mathbb{N}]^k$ such that

$$\Big\| \sum_{j=1}^{k} x_{(n_1,\dots,n_j)} \Big\| \leq \gamma k,$$

one can prove as we did before that, for all $k, l \in \mathbb{N}$, $\gamma_{kl}(X) \leq \gamma_k(X)\gamma_l(X)$. Therefore, in view of Lemma 4.4.4, if $\gamma_k(X) < 1$ for some $k \in \mathbb{N}$, then there exists $p' \in [1, \infty)$ such that $\gamma_k(X) = \frac{1}{k^{1/p'}}$ and X is q-AUS-able for all $q < p$, $\frac{1}{p} + \frac{1}{p'} = 1$. Thus, the existence of such a function φ is not weaker than asking for asymptotic uniform smoothness up to renorming.

4.6.2 A remark concerning upper ℓ_p tree estimates

If X has a weak unconditional asymptotic structure, then, by definition, there exists a constant C such that for every $k \in \mathbb{N}$, for every weakly null tree $(x_{\overline{n}})_{\overline{n}\in[\mathbb{N}]^{\leq k}} \subset B_X$, one can find $\overline{n} \in [\mathbb{N}]^k$ so that

$$\Big\| \sum_{j=1}^{k} \varepsilon_j x_{(n_1,\cdots,n_j)} \Big\| \leq C \Big\| \sum_{j=1}^{k} x_{(n_1,\cdots,n_j)} \Big\|$$

for all $(\varepsilon_1, \cdots, \varepsilon_k) \in \{\pm 1\}^k$.
It follows that if X has asymptotic Rademacher type $p \in (1, \infty)$ and X has a weak unconditional asymptotic structure, then X has A_p (see Chapter 1). In particular, X is q-AUS-able for all $q \in (1, p)$.

4.6.3 Another difference between the local and asymptotic settings

In the local setting, if a Banach space X uniformly contains the ℓ_∞^n, $n \in \mathbb{N}$, then it uniformly contains all the finite dimensional spaces. However, if the ℓ_∞^n, $n \in \mathbb{N}$, are uniformly in the asymptotic structure of X, it does not mean that all the finite dimensional spaces can be uniformly found in its asymptotic structure. Indeed, if you consider any asymptotic-c_0 space, then ℓ_1 cannot be asymptotically finitely representable in it by Theorem 4.2.9 and 4.2.2.

4.6.4 Open questions

We conclude this chapter with the following questions.

Problem 4.6.1. Does $(2,\varepsilon)$ asymptotic convexity for some $\varepsilon > 0$ imply asymptotic uniform smoothness up to renorming?

Problem 4.6.2. Can one find an asymptotic notion of K-convexity that would be equivalent to asymptotic B-convexity?

Problem 4.6.3. Does a reflexive asymptotic B-convex Banach space have a concentration property from Chapter 2?

Appendix A

Distance between c_0 and some $C(K)$

The goal of this appendix is to give a different proof of the following fact due to Candido and Galego in 2012 [22].

Theorem A.0.1 ([22])**.** *For every $n \in \mathbb{N}$, $d_{BM}(C([0,\omega^n]), c_0) \geq 2n+1$.*

To prove this lower bound, Candido and Galego use results coming from measure theory and topology in order to generalize Cambern's proof of the fact that $d_{BM}(c, c_0) = 3$ (see [21]). As for us, we are going to use asymptotic considerations of the spaces at stake.

To see the idea behind our proof, we start by proving the following proposition that will be generalized soon after.

Proposition A.0.2. *Let Y be a subspace of c_0. Then $d_{BM}(c, Y) \geq 3$.*

Proof. Let $T : Y \to c$ be a linear isomorphism such that $\|T^{-1}\| = 1$.
First, note that $x_n = T^{-1}(e_n - s_n)$ is in B_Y, where $u = (1)_{n\in\mathbb{N}}$ and $s_n = u - \sum_{j=1}^{n} e_j$ for $n \in \mathbb{N}$. Next, we have

$$\begin{aligned}
3 &\leq \liminf |(e_n^* - \delta_\omega)(e_n - s_n)| + |\delta_\omega(u)| = \liminf |(e_n^* - \delta_\omega) \circ T(x_n)| + |\delta_\omega \circ T(T^{-1}(u))| \\
&\leq \liminf \|(e_n^* - \delta_\omega) \circ T\| + \|\delta_\omega \circ T\| \;\text{ since } (x_n) \subset B_Y \text{ and } T^{-1}(u) \in B_Y \\
&\leq \liminf \|(e_n^* - \delta_\omega) \circ T + \delta_\omega \circ T\| \;\text{ since } ((e_n^* - \delta_\omega) \circ T) \subset Y^* \text{ is weak}^*\text{-null and } Y \subset c_0 \\
&= \liminf \|e_n^* \circ T\| \leq \|T\|.
\end{aligned}$$

The conclusion follows. □

Let X be a Banach space. For every weak* compact set $K \subset X^*$ and $\varepsilon > 0$, we write

$$l_\varepsilon(K) = \{x^* \in K;\ \exists (x_n^*) \subset K,\ \omega^* - \lim x_n^* = x^* \text{ and } \|x_n^* - x^*\| \geq \varepsilon\ \forall n\}.$$

We can now state the following result, which easily implies A.0.1.

Proposition A.0.3. *Let Y be a subspace of c_0. For every Banach space X, we have*

$$d_{BM}(Y, X) \geq \max_{\substack{k\in\mathbb{N}, \varepsilon_1, \cdots, \varepsilon_k > 0 \\ l_{\varepsilon_1} l_{\varepsilon_2} \cdots l_{\varepsilon_k}(B_{X^*}) \neq \varnothing}} \; \max_{x^* \in l_{\varepsilon_1} l_{\varepsilon_2} \cdots l_{\varepsilon_k}(B_{X^*})} \|x^*\| + \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_k.$$

Proof. Let $k \in \mathbb{N}$, $\varepsilon_1, \ldots, \varepsilon_k > 0$, $x^* \in l_{\varepsilon_1} l_{\varepsilon_2} \ldots l_{\varepsilon_k}(B_{X^*})$. Let $\delta > 0$, $T : Y \to X$ such that $\|T^{-1}\| = 1$.
We can find $x \in B_X$ and sequences $(x^*_{n_1})_{n_1 \in \mathbb{N}}, \ldots, (x^*_{n_1,\ldots,n_k})_{n_1,\ldots,n_k \in \mathbb{N}} \subset B_{X^*}$, $(x_{n_1})_{n_1 \in \mathbb{N}}, \ldots, (x_{n_1,\ldots,n_k})_{n_1,\ldots,n_k \in \mathbb{N}} \subset S_X$ such that $x^*(x) \geq \|x^*\| - \delta$, $(x^*_{n_1})_{n_1 \in \mathbb{N}}$ weak* converges to x^*, $(x^*_{n_1} - x^*)(x_{n_1}) \geq \varepsilon_1 - \delta$ for every $n_1 \in \mathbb{N}$, $(x^*_{n_1,\ldots,n_{j-1},n_j})_{n_j \in \mathbb{N}}$ weak* converges to $x^*_{n_1,\ldots,n_{j-1}}$, $(x^*_{n_1,\ldots,n_{j-1},n_j} - x^*_{n_1,\ldots,n_{j-1}})(x_{n_1,\ldots,n_{j-1},n_j}) \geq \varepsilon_j - \delta$ for every $j \in \{2, \cdots, k\}$ and every $(n_1, \cdots, n_j) \in \mathbb{N}^j$. Now, for all $(n_1, \cdots, n_j) \in \mathbb{N}^j$, we have

$$\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_k + \|x^*\| - (k+1)\delta \leq (x^*_{n_1} - x^*)(x_{n_1}) + \sum_{j=2}^{k} (x^*_{n_1,\cdots,n_j} - x^*_{n_1,\cdots,n_{j-1}})(x_{n_1,\cdots,n_j}) + x^*(x)$$

$$= (x^*_{n_1} - x^*) \circ T(T^{-1}x_{n_1}) + \sum_{j=2}^{k} (x^*_{n_1,\cdots,n_j} - x^*_{n_1,\cdots,n_{j-1}}) \circ T(T^{-1}x_{n_1,\cdots,n_j}) + x^* \circ T(T^{-1}x)$$

$$\leq \|(x^*_{n_1} - x^*) \circ T\| + \sum_{j=2}^{k} \|(x^*_{n_1,\cdots,n_j} - x^*_{n_1,\cdots,n_{j-1}}) \circ T\| + \|x^* \circ T\|$$

Since Y is a subset of c_0, we can find $(n_1, \cdots, n_k) \in \mathbb{N}^k$ so that

$$\|(x^*_{n_1} - x^*) \circ T\| + \|x^* \circ T\| \leq \|x^*_{n_1} \circ T\| + \delta$$

and

$$\|(x^*_{n_1,\cdots,n_j} - x^*_{n_1,\cdots,n_{j-1}}) \circ T\| + \|x^*_{n_1,\cdots,n_{j-1}} \circ T\| \leq \|x^*_{n_1,\cdots,n_j} \circ T\| + \delta$$

for every $j \in \{2, \cdots, k\}$. This gives us

$$\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_k + \|x^*\| - (k+1)\delta \leq \|x^*_{n_1,\cdots,n_k} \circ T\| + k\delta \leq \|T\| + k\delta.$$

The result follows since we can take δ as small as we want. □

Remark A.0.4. Let us note that if Y satisfies $\liminf \|y^* + y^*_n\|^q + \|y^*\|^q \leq \liminf \|y^*_n\|^q$ for every weak* null sequence $(y^*_n) \subset Y^*$ and every $y^* \in Y^*$, then

$$d_{BM}(Y, X) \geq \max_{\substack{n \in \mathbb{N}, \varepsilon_1, \cdots, \varepsilon_n > 0 \\ l_{\varepsilon_1} l_{\varepsilon_2} \cdots l_{\varepsilon_n}(B_{X^*}) \neq \varnothing}} \max_{x^* \in l_{\varepsilon_1} l_{\varepsilon_2} \cdots l_{\varepsilon_n}(B_{X^*})} (\|x^*\|^q + \varepsilon_1^q + \varepsilon_2^q + \cdots + \varepsilon_n^q)^{1/q}.$$

In their paper, Candido and Galego prove in fact that the Banach-Mazur distance between $C([0, \omega^n])$ and c_0 is equal to $2n+1$ for every $n \in \mathbb{N}$ (see [22]). However, the value of the Lipschitz distance seems unknown (we recall that the Lipschitz distance between two metric spaces (M, d) and (N, δ) is the infinimum of $\mathrm{Lip}(f)\mathrm{Lip}(f^{-1})$ over all Lipschitz isomorphisms $f : M \to N$).
By combining Theorem 6.3 of [32] and classical links between the Szlenk index and the modulus of weak* asymptotic uniform convexity, one can find a universal constant $C > 0$ such that the Lipschitz distance between c_0 and $C([0, \omega^n])$ is bigger that $C\sqrt{n}$ for every $n \in \mathbb{N}$. This is a quantitative version of the fact that c_0 and $C([0, \omega^\omega])$ are not Lipschitz equivalent. Let us indicate some details.
Let $X = C([0, \omega^n])$, $f : c_0 \to X$ a Lipschitz bijection such that $\mathrm{Lip}(f) \leq 1$ and $\mathrm{Lip}(f^{-1}) \leq$

M. By Theorem Theorem 6.3 of [32], there exists a norm $|.|$ on X such that $\|.\|_X \leq |.| \leq M\|.\|_X$ and

$$\forall t \in [0,1],\ \overline{\delta}^*_{|.|}(t) \geq \overline{\delta}^*_{c_0}\left(\frac{t}{4M}\right) = \frac{t}{4M}.$$

Furthermore, there exists $C \geq 1$ such that

$$\forall \varepsilon \in (0,1),\ Sz((X,|.|),\varepsilon) \leq C\left(\overline{\delta}^*_{|.|}(\frac{\varepsilon}{C})\right)^{-1}.$$

Since $Sz((X,|.|),1/2M) \geq n+1$, the result follows from

$$n+1 \leq Sz((X,|.|),1/2M) \leq C\left(\overline{\delta}^*_{|.|}(\frac{1}{2MC})\right)^{-1} \leq 8M^2C^2.$$

In order to give a similar quantitative version of the fact that c_0 and $C([0,\omega^\omega])$ are not coarse Lipschitz equivalent, we define a coarse Lipschitz écart between Banach spaces. If X and Y are two Banach spaces, we call coarse Lipschitz écart between X and Y, the infinimum of A/C where $A, C > 0$ are such that one can find a map $f : X \to Y$ and constants $C, D > 0$ so that

$$\forall x,y \in X,\ C\|x-y\| - D \leq \|f(x)-f(y)\| \leq A\|x-y\| + B.$$

Now, by applying this time Theorem 6.5 from [32], one gets another universal constant $C > 0$ such that the coarse Lipschitz écart between c_0 and $C([0,\omega^n])$ is bigger that $C\sqrt{n}$ for every $n \in \mathbb{N}$.

Appendix B

Some non-linear indices

The goal of this appendix is to generalize some of the results of [10] by defining similar non-linear indices that "measure" how present are Lipschitz or coarse-Lipschitz copies of certain spaces into a given one.

Let us briefly describe the content of this appendix. First, we will recall some definitions about trees and vines. Next, we will give two necessary and sufficient conditions for a Polish space to contain a Lipschitz copy of ℓ_p, $1 \leq p < \infty$, in order to deduce that ℓ_1 Lipschitz embeds into a Banach space if and only if this space contains bi-Lipschitz copies of all reflexive asymptotic-ℓ_1 spaces. The second section will be dedicated to the proof of similar results in a coarse-Lipschitz setting. We will conclude this appendix by giving a characterization of the index of Lipschitz containment of ℓ_p being bigger than ω and by raising a few questions.

Before proving our first propositions, in order to ease the reading, we recall some definitions about trees. First, if X is a set, we will call *tree over* X a collection T of finite sequences of X such that $(x_1, \cdots, x_n)$ is in T as soon as $(x_1, \cdots, x_{n+1})$ is in T for some $x_{n+1} \in X$. A tree will be said *well-founded* if it does not contain any infinite branch, where an infinite branch would be a sequence $(x_n)_{n=1}^{\infty}$ in X satisfying $(x_1, \cdots, x_n) \in T$ for every $n \in \mathbb{N}$. We will now define the order of a tree T over a set X, based on the following transfinite derivation

$$\begin{aligned}
&T^0 = T \\
&T^{\alpha+1} = \{(x_1, \cdots, x_n);\ (x_1, \cdots, x_n, x_{n+1}) \in T^{\alpha}\}, \text{ for any ordinal } \alpha \text{ (we allow } n = 0) \\
&T^{\beta} = \bigcap_{\alpha<\beta} T^{\alpha}, \text{ for any limit ordinal } \beta.
\end{aligned}$$

We define the *order of* T, denoted $\mathrm{o}(T)$, to be the least ordinal number such that $T^{\mathrm{o}(T)} = \varnothing$, with the convention $\mathrm{o}(T) = \infty$ if there is no such ordinal. One can note that $\mathrm{o}(T) < \infty$ if T is well-founded.

Let us now define our indices of non-linear presence. We start with the Lipschitz setting.

B.1 Lipschitz setting

First, let us recall the definitions about vines, introduced in [10].
Let $\mathbb{E} = \mathbb{Q}$ or $\mathbb{E} = \mathbb{Z}$. For all $G \in [\mathbb{N}]^{<\omega}$, we denote

$$[\mathbb{E}, G] = \{f : \mathbb{N} \to \mathbb{E}; \mathrm{Supp}(f) \subset G\}$$

that can be seen as a subset of $c_{00}(\mathbb{N}, \mathbb{R})$.
Given a set X, an $\mathbb{E}$-*bunch over* X will be an element of the form $\chi = (x_f)_{f\in[\mathbb{E},G]}$ in $X^{[\mathbb{E},G]}$. We can endow the set of $\mathbb{E}$-bunches over X with a partial order as follows. For $\chi = (x_f)_{f\in[\mathbb{E},F]}$ and $\psi = (y_f)_{f\in[\mathbb{E},G]}$, we write $\chi \preceq \psi$ if F is an initial segment of G and $y_f = x_f$ for every $f \in [\mathbb{E}, F]$. As for trees, we will say that a set $\mathscr{V}$ of $\mathbb{E}$-bunches over X is an $\mathbb{E}$-*vine over* X if the set $[\psi \preceq \chi]$ is a subset of $\mathscr{V}$ for every $\chi \in \mathscr{V}$ and $\mathscr{V}$ will be called *well-founded* if the tree $(\mathscr{V}, \preceq)$ is well-founded, *i.e* it contains no infinite totally ordered subset. Let us also define the ordinal index of a vine. For a vine $\mathscr{V}$, we put

$$\mathscr{V}^{(1)} = \mathscr{V} \setminus \{\chi \in \mathscr{V};\ \chi \text{ is } \preceq\text{-maximal}\},$$

and, recursively

$$\mathscr{V}^{(\alpha+1)} = (\mathscr{V}^{(\alpha)})^{(1)}, \text{for any ordinal } \alpha$$
$$\mathscr{V}^{(\beta)} = \bigcap_{\alpha<\beta} \mathscr{V}^{(\alpha)}, \text{ for any limit ordinal } \beta.$$

Then, the *ordinal index of* $\mathscr{V}$, denoted $\mathrm{o}(\mathscr{V})$, is defined to be the least ordinal α such that $\mathscr{V}^{(\alpha)} = \varnothing$, which is well-defined if $\mathscr{V}$ is well-founded.

Now, for any norm $\|.\|$ on $c_{00}(\mathbb{N}, \mathbb{R})$, any $C > 0$ and any metric space (M, d), we denote

$$\mathscr{V}_{\|.\|}(M, \mathbb{E}, C) = \bigcup_{G\in[\mathbb{N}]^{<\omega}} \left\{(x_f)_{f\in[\mathbb{E},G]} \in M^{[\mathbb{E},G]}; \forall f, g \in [\mathbb{E}, G], \frac{1}{C}\|f - g\| \leq d(x_f, x_g) \leq C\|f - g\|\right\}$$

where $c_{00}(\mathbb{N}, \mathbb{E})$ is seen as a subset of $c_{00}(\mathbb{N}, \mathbb{R})$. We will also allow ourselves the following notation, even though $\mathbb{E}$ is not a vector space:

$$\mathrm{I}^{\mathrm{Lip}}_{(c_{00}(\mathbb{N},\mathbb{E}),\|.\|)}(M) = \sup\{\mathrm{o}(\mathscr{V}_{\|.\|}(M, \mathbb{E}, C)), C > 0\}.$$

When $\mathbb{E} = \mathbb{Q}$ and $\|.\| = \|.\|_p$, $1 \leq p \leq \infty$, we will write $\mathrm{I}^{\mathrm{Lip}}_{\ell_p}(M)$ instead of $\mathrm{I}^{\mathrm{Lip}}_{(c_{00}(\mathbb{N},\mathbb{E}),\|.\|_p)}(M)$ if $p < \infty$ and $\mathrm{I}^{\mathrm{Lip}}_{c_0}(M)$ instead of $\mathrm{I}^{\mathrm{Lip}}_{(c_{00}(\mathbb{N},\mathbb{Q}),\|.\|_p)}(M)$ if $p = \infty$.

Let us start with the following proposition.

Proposition B.1.1. *Let* (M, d) *be a Polish space. We have:*

$$(c_{00}(\mathbb{N}, \mathbb{E}), \|.\|) \underset{L}{\hookrightarrow} M \iff I^{Lip}_{(c_{00}(\mathbb{N},\mathbb{E}),\|.\|)}(M) \geq \omega_1.$$

Proof. Assume $\psi : c_{00}(\mathbb{N}, \mathbb{E}) \to M$ satisfies:

$$\forall x, y \in c_{00}(\mathbb{N}, \mathbb{E}),\ \frac{1}{C}\|x - y\| \leq d(\psi(x), \psi(y)) \leq C\|x - y\|.$$

for a $C \geq 1$.
For all $G \in [\mathbb{N}]^{<\omega}$, $f \in [\mathbb{E}, G]$, we put $x_f = \psi\left(\sum_{i\in G} f(i)e_i\right)$. We can see that

$$\{(x_f)_{f\in[\mathbb{E},G]}; G \in [\mathbb{N}]^{<\omega}\} \subset \mathscr{V}_{\|.\|}(M, \mathbb{E}, C)$$

hence $\mathrm{I}^{\mathrm{Lip}}_{c_{00}(\mathbb{N},\mathbb{E})}(M) \geq \omega_1$.
Assume now $\mathrm{I}^{\mathrm{Lip}}_{c_{00}(\mathbb{N},\mathbb{E})}(M) = \omega_1$.
For all countable ordinal α, there exists $C_\alpha > 0$ such that $\mathrm{o}(\mathscr{V}_{\|.\|}(M, \mathbb{E}, C_\alpha)) \geq \alpha$. We can find $C > 0$ and an uncountable set $U \subset [1, \omega_1)$ such that $C_\alpha \leq C$ and so

$$\mathrm{o}(\mathscr{V}_{\|.\|}(M, \mathbb{E}, C)) \geq \mathrm{o}(\mathscr{V}_{\|.\|}(M, \mathbb{E}, C_\alpha)) \geq \alpha$$

for all $\alpha \in U$. As a consequence, $\mathscr{V}_{\|.\|}(M, \mathbb{E}, C)$ is ill-founded, *i.e* there exists a strictly increasing sequence of integers (k_m) and, for $m \in \mathbb{N}\cup\{0\}$, $\chi_m = (x_f^{(m)}; f \in [\mathbb{E}, \{k_1, \cdots, k_m\}]) \in \mathscr{V}_{\|.\|}(M, \mathbb{E}, C)$ so that $\chi_0 \preceq \chi_1 \preceq \chi_2 \cdots$. This means that for every finitely supported $f : \{k_1, k_2, k_3, \cdots\} \to \mathbb{E}$, there is an $x_f \in M$ such that $\chi_m = (x_f; f \in [\mathbb{E}, \{k_1, \cdots, k_m\}])$ for $m \in \mathbb{N}$. It remains to note that the map $\psi : (c_{00}(\mathbb{N}, \mathbb{E}), \|.\|) \to M$ defined by

$$\psi((s_j)) = x_f \text{ where } f : \{k_1, k_2, \cdots\} \to \mathbb{E}, \text{ is defined by } f(k_j) = s_j$$

is a Lipschitz embedding to conclude. □

Before proving our generalizations, let us recall the definition of the Schreier sets $(S_\alpha)_{\alpha<\omega_1}$. First, if A and B are finite subsets of $\mathbb{N}$, we write $n \leq A < B$ if $n \leq \min(A) \leq \max(A) < \min(B)$. Now, if α is a countable ordinal, we denote by $S_\alpha \subset [\mathbb{N}]^{<\omega}$ the Schreier family of order α, defined recursively as follows:

$$S_0 = \{\{n\}; n \in \mathbb{N}\},$$
$$S_{\alpha+1} = \Big\{\bigcup_{j=1}^{n} E_j; E_j \in S_\alpha \text{ for all } 1 \leq j \leq n \text{ and } n \leq E_1 < E_2 < \cdots < E_n\Big\},$$
$$S_\beta = \{A \in [\mathbb{N}]^{<\omega}; \exists n \in \mathbb{N}, n \leq A, A \in S_{\alpha_n}\}$$

if β is a limit ordinal, and $(\alpha_n) \subset [0, \beta)$ is a (fixed) sequence which increases to β (the choice of (α_n) is irrelevant).
These families naturally generate trees on $\mathbb{N}$ by setting

$$T(S_\alpha) = \{(n_1, \ldots, n_k); \{n_i\}_{i=1}^k \in S_\alpha\}$$

for every ordinal α. With a transfinite induction, one can check that its order is $\mathrm{o}(T(S_\alpha)) = \omega^\alpha + 1$ (see [3]).
Let us denote $(S_\alpha(\mathbb{E}), d_{\|.\|})$ the subset of $c_{00}(\mathbb{N}, E)$

$$S_\alpha(\mathbb{E}) = \{\sum_{i\in G} c_i e_i,\ G \in S_\alpha, c_i \in \mathbb{E}\}$$

endowed with $\|.\|$ for every ordinal α.

Proposition B.1.2. *Let (M, d) be a Polish space. We have:*

$$(c_{00}(\mathbb{N}, \mathbb{E}), \|.\|) \underset{L}{\hookrightarrow} M \iff \forall\alpha,\ (S_\alpha(\mathbb{E}), d_{\|.\|}) \underset{L}{\hookrightarrow} M.$$

Proof. The direct implication is clear. Reciprocally, let us assume that M admits bi-Lipschitz embeddings of $(S_\alpha(\mathbb{E}), d_{\|.\|})$ for every ordinal α. After an eventuel extraction argument, we can find a constant $C > 0$, an uncountable set $A \subset [0, \omega_1)$, and maps $F_\alpha : (S_\alpha(\mathbb{E}), d_{\|.\|}) \to M$ for all $\alpha \in A$ such that

$$\frac{1}{C}\|f - g\| \leq d(F_\alpha(f), F_\alpha(g)) \leq C\|f - g\|$$

for all $f, g \in S_\alpha(\mathbb{E})$ and $\alpha \in A$.
Thus, as in [10], $\mathscr{V}_{\|.\|}(M, \mathbb{E}, C)$ has ordinal index at least $\mathrm{o}(S_\alpha) = \omega^\alpha + 1$, for all $\alpha \in A$, so $\mathrm{I}^{\mathrm{Lip}}_{(c_{00}(\mathbb{N},\mathbb{E}),\|.\|)}(M) \geq \omega_1$. The result follows from the previous proposition. □

Let us note a direct consequence.

Proposition B.1.3. *Let (M, d) be a Polish space, $1 \leq p < \infty$. Then*

$$\ell_p \underset{L}{\hookrightarrow} M \iff I^{Lip}_{\ell_p}(M) \geq \omega_1 \iff \forall \alpha,\ (S_\alpha(\mathbb{Q}), d_{\|.\|_p}) \underset{L}{\hookrightarrow} M.$$

The separable reflexive Banach spaces T_α, which are the higher order Tsirelson spaces, are asymptotic-ℓ_1 and have a 1-unconditional basis (u_i) with the property that for any ordinal α and any $G \in S_\alpha$, the sequence $(u_i)_{i \in G}$ is 2-equivalent to the unit vector basis of $\ell_1^{|G|}$ (see [86]). It follows that the natural embedding of $(S_\alpha(\mathbb{E}), d_1)$ in T_α is a 4-Lipschitz isomorphism. Thus, we have:

Proposition B.1.4. *If a Banach space Y contains bi-Lipschitz copies of every reflexive asymptotic-ℓ_1 space, then ℓ_1 Lipschitz embeds into Y.*

In the case of c_0, without further arguments, we can already deduce the following.

Proposition B.1.5. *If a Banach space Y contains coarse-Lipschitz copies of every reflexive asymptotic-c_0 space, then c_0 coarse-Lipschitz embeds into Y.*

Proof. Let Y such a Banach space. Then Y contains a coarse-Lipschitz copy of $(S_\alpha(\mathbb{Q}), d_\infty)$ for every α, thus Y contains a Lipschitz copy of every $(S_\alpha(\mathbb{Z}), d_\infty)$. Hence $(c_{00}(\mathbb{N}, \mathbb{Z}), \|.\|_\infty)$ Lipschitz embeds into Y, which allows us to conclude that c_0 coarse-Lipschitz embeds into Y (see Proposition B.2.1 for a more general result). □

B.2 Coarse Lipschitz setting

The previous results can be extended to the coarse Lipschitz setting via minor adjustments. Let us see how.

For any norm $\|.\|$ on $c_{00}(\mathbb{N}, \mathbb{R})$, any $C > 0$ and any metric space (M, d), we denote

$$\mathscr{V}^{cL}_{\|.\|}(M, \mathbb{E}, C) = \bigcup_{G \in [\mathbb{N}]^{<\omega}} \Big\{ (x_f)_{f \in [\mathbb{E}, G]} \in M^{[\mathbb{E}, G]}; \forall f, g \in [\mathbb{E}, G], \frac{1}{C}\|f - g\| - C \leq d(x_f, x_g) \leq C\|f - g\| + C \Big\}$$

and, as before, we will allow ourselves the notation

$$\mathrm{I}^{\mathrm{cL}}_{(c_{00}(\mathbb{N},\mathbb{E}),\|.\|)}(M) = \sup\{\mathrm{o}(\mathscr{V}^{cL}_{\|.\|}(M,\mathbb{E},C)), C>0\}.$$

As earlier, for $p \in [1,\infty)$, we will use the notation $\mathrm{I}^{\mathrm{cL}}_{\ell_p}(M)$ instead $\mathrm{I}^{\mathrm{cL}}_{(c_{00}(\mathbb{N},\mathbb{Q}),\|.\|_p)}(M)$, as well as $\mathrm{I}^{\mathrm{cL}}_{c_0}(M)$ instead $\mathrm{I}^{\mathrm{cL}}_{(c_{00}(\mathbb{N},\mathbb{Q}),\|.\|_\infty)}(M)$. Let us immediately note the following.

Proposition B.2.1. *For any metric space (M,d), we have:*

$$I^{cL}_{c_0}(M) = I^{Lip}_{(c_{00}(\mathbb{N},\mathbb{Z}),\|.\|_\infty)}(M).$$

Proof. Let $C \geq 1$. Let us start by proving that $\mathrm{o}(\mathscr{V}_{\|.\|_\infty}(M,\mathbb{Z},C)) \leq \mathrm{o}(\mathscr{V}^{cL}_{\|.\|_\infty}(M,\mathbb{Q},C))$. Let $(x_f)_{f\in[\mathbb{Z},G]} \in \mathscr{V}_{\|.\|_\infty}(M,\mathbb{Z},C)$, $G \in [\mathbb{N}]^{<\omega}$. We associate to every $f \in [\mathbb{Q},G]$ an element $\tilde{f} \in [\mathbb{Z},G]$ so that $\|f-\tilde{f}\|_\infty \leq \frac{1}{2}$. Next, for every $f \in [\mathbb{Q},G]$, we define $\tilde{x}_f = x_{\tilde{f}}$. To get the inequality, it is enough to note that

$$d(\tilde{x}_f,\tilde{x}_g) = d(x_{\tilde{f}},x_{\tilde{g}}) \leq C\|\tilde{f}-\tilde{g}\|_\infty \leq C\|f-g\|_\infty + C$$

and

$$d(\tilde{x}_f,\tilde{x}_g) = d(x_{\tilde{f}},x_{\tilde{g}}) \geq \frac{1}{C}\|\tilde{f}-\tilde{g}\|_\infty \geq \frac{1}{C}\|f-g\|_\infty - \frac{1}{C} \geq \frac{1}{C}\|f-g\|_\infty - C$$

for every $f,g \in [\mathbb{Q},G]$.
Now, let $(x_f)_{f\in[\mathbb{Q},G]} \in \mathscr{V}^{cL}_{\|.\|_\infty}(M,\mathbb{Q},C)$, $G \in [\mathbb{N}]^{<\omega}$. There exist $\theta \in [1,\infty)\cap\mathbb{Q}$ and $D \geq 1$ (that only depend on C) so that

$$\forall f,g \in [\mathbb{Q},G],\ \|f-g\|_\infty \geq \theta \implies \frac{1}{D}\|f-g\|_\infty \leq d(x_f,x_g) \leq D\|f-g\|_\infty.$$

For every $f \in [\mathbb{Z},G]$, we define $\tilde{f} = \theta f \in [\mathbb{Q},G]$ and $\tilde{x}_f = x_{\tilde{f}}$. For every $f \neq g \in [\mathbb{Z},G]$, $\|\tilde{f}-\tilde{g}\|_\infty \geq \theta$ so

$$\frac{\theta}{D}\|f-g\|_\infty = \frac{1}{D}\|\tilde{f}-\tilde{g}\|_\infty \leq d(\tilde{x}_f,\tilde{x}_g) \leq D\|\tilde{f}-\tilde{g}\|_\infty = \theta D\|f-g\|_\infty.$$

As a consequence, we get $\mathrm{o}(\mathscr{V}_{\|.\|_\infty}(M,\mathbb{Z},\theta D)) \geq \mathrm{o}(\mathscr{V}^{cL}_{\|.\|_\infty}(M,\mathbb{Q},C))$, which finishes the proof. □

Proposition B.2.2. *Let (M,d) be a Polish space. We have:*

$$(c_{00}(\mathbb{N},\mathbb{E}),\|.\|) \underset{cL}{\hookrightarrow} M \iff I^{cL}_{(c_{00}(\mathbb{N},\mathbb{E}),\|.\|)}(M) \geq \omega_1.$$

Proof. Assume $\psi : c_{00}(\mathbb{N},\mathbb{E}) \to M$ satisfies:

$$\forall x,y \in c_{00}(\mathbb{N},\mathbb{E}),\ \frac{1}{C}\|x-y\| - C \leq d(\psi(x),\psi(y)) \leq C\|x-y\| + C.$$

for some $C \geq 1$.
For all $G \in [\mathbb{N}]^{<\omega}$, $f \in [\mathbb{E},G]$, we put $x_f = \psi\left(\sum_{i\in G} f(i)e_i\right)$. We can see that

$$\{(x_f)_{f\in[\mathbb{E},G]}; G \in [\mathbb{N}]^{<\omega}\} \subset \mathscr{V}^{cL}_{\|.\|}(M,\mathbb{E},C)$$

hence $\mathrm{I}^{\mathrm{cL}}_{c_{00}(\mathbb{N},\mathbb{E})}(M) \geq \omega_1$.
Assume now $\mathrm{I}^{\mathrm{Lip}}_{c_{00}(\mathbb{N},\mathbb{E})}(M) = \omega_1$.
For all countable ordinal α, there exists $C_\alpha > 0$ such that $\mathrm{o}(\mathscr{V}^{cL}_{\|.\|}(M,\mathbb{E},C_\alpha)) \geq \alpha$. We can find $C > 0$ and an uncountable set $U \subset [1,\omega_1)$ such that $C_\alpha \leq C$ and so

$$\mathrm{o}(\mathscr{V}^{cL}_{\|.\|}(M,\mathbb{E},C)) \geq \mathrm{o}(\mathscr{V}^{cL}_{\|.\|}(M,\mathbb{E},C_\alpha)) \geq \alpha$$

for all $\alpha \in U$. As a consequence, $\mathscr{V}^{cL}_{\|.\|}(M,\mathbb{E},C)$ is ill-founded, *i.e* there exists a strictly increasing sequence of integers (k_m) and, for $m \in \mathbb{N}\cup\{0\}$, $\chi_m = (x^{(m)}_f; f \in [\mathbb{E},\{k_1,\cdots,k_m\}]) \in \mathscr{V}^{cL}_{\|.\|}(M,\mathbb{E},C)$ so that $\chi_0 \preceq \chi_1 \preceq \chi_2 \cdots$. This means that for every finitely supported $f : \{k_1,k_2,k_3,\cdots\} \to \mathbb{E}$, there is an $x_f \in M$ so that $\chi_m = (x_f; f \in [\mathbb{E},\{k_1,\cdots,k_m\}])$ for $m \in \mathbb{N}$. It remains to note that the map $\psi : (c_{00}(\mathbb{N},\mathbb{E}),\|.\|) \to M$ defined by

$$\psi((s_j)) = x_f \text{ where } f : \{k_1,k_2,\cdots\} \to \mathbb{E}, \text{ is defined by } f(k_j) = s_j$$

is a coarse-Lipschitz embedding to conclude. □

With an almost identical proof as for Proposition B.1.2, we get:

Proposition B.2.3. *Let (M,d) be a Polish metric space. We have:*

$$(c_{00}(\mathbb{N},\mathbb{E}),\|.\|) \underset{cL}{\hookrightarrow} M \iff \forall\alpha,\ (S_\alpha(\mathbb{E}),d_{\|.\|}) \underset{cL}{\hookrightarrow} M.$$

As before, let us note those two direct consequences.

Proposition B.2.4. *Let (M,d) be a Polish metric space, $1 \leq p < \infty$. Then*

$$\ell_p \underset{cL}{\hookrightarrow} M \iff I^{cL}_{\ell_p}(M) \geq \omega_1.$$

Proposition B.2.5. *If a Banach space Y contains coarse-Lipschitz copies of every reflexive asymptotic-ℓ_1 space, then ℓ_1 coarse-Lipschitz embeds into Y.*

B.3 Quantitative estimates and open problems

In this final section, we give a necessary and sufficient condition for $\mathrm{I}^{\mathrm{Lip}}_{\ell_p}(X)$, $\mathrm{I}^{\mathrm{Lip}}_{c_0}(X)$, $\mathrm{I}^{\mathrm{cL}}_{\ell_p}(X)$, $\mathrm{I}^{\mathrm{cL}}_{c_0}(X)$ to be bigger than ω. The result is the following.

Proposition B.3.1. *Let X be a Banach space, $1 \leq p < \infty$. The following are equivalent:*

(i) $I^{Lip}_{\ell_p}(X) > \omega$;

(ii) $I^{cL}_{\ell_p}(X) > \omega$;

(iii) ℓ_p *is finitely representable in* X.

Similarly, $I^{Lip}_{c_0}(X) > \omega$ if and only if $I^{cL}_{c_0}(X) > \omega$ if and only if c_0 is finitely representable in X.

Proof. $*$ The implication $(i) \implies (ii)$ is clear.
$*$ Assume $\mathrm{I}^{\mathrm{cL}}_{\ell_p}(X) > \omega$. There exists $C > 0$ such that

$$\mathrm{o}(\mathcal{V}(X, \mathbb{Q}, C)) > \omega \text{ where } \mathcal{V}(X, \mathbb{Q}, C) := \mathscr{V}^{\mathrm{cL}}_{\|\cdot\|_p}(X, \mathbb{Q}, C)$$

Then $\bigcap_{n\in\mathbb{N}} \mathcal{V}^{(n)}(X, \mathbb{Q}, C) \neq \varnothing$. Thus we can find $G \in [\mathbb{N}]^{<\omega}$, $(x_f)_{f\in[\mathbb{Q},G]} \in \bigcap_{n\in\mathbb{N}} \mathcal{V}^{(n)}(X, \mathbb{Q}, C)$. Let $k \in \mathbb{N}$. We can find $s_1 < \cdots < s_k \notin G$ (we note $G' = G \cup \{s_1, \cdots, s_k\}$) and $x_f \in X$ for all $f \in [\mathbb{Q}, G'] \setminus [\mathbb{Q}, G]$, such that $\{(x_f)_{f\in[\mathbb{Q},G']}\} \subset \mathcal{V}(X, \mathbb{Q}, C)$.
Then the map $\varphi : \begin{cases} \mathbb{Q}^k & \to X \\ (q_1, \cdots, q_k) & \mapsto x_{q_1 e_{s_1} + \cdots + q_k e_{s_k}} \end{cases}$ satisfies

$$\forall q = (q_i)_{i=1}^k, q' = (q'_i)_{i=1}^k \in \mathbb{Q}^k,\ \frac{1}{C}\|q - q'\|_p - C \leq \|\varphi(q) - \varphi(q')\| \leq C\|q - q'\| + C.$$

By density, we get that the ℓ_p^n, $n \in \mathbb{N}$ uniformly coarse-Lipschitz embed into X. Ribe's Theorem asserts that ℓ_p is finitely representable in X.
$*$ Finally, let us prove that $(iii) \implies (i)$. Assume that ℓ_p is finitely representable in X. We will show that $0 \in \bigcap_{n\in\mathbb{N}} \mathscr{V}^{(n)}_{\|\cdot\|_p}(X, \mathbb{Q}, C)$ for some $C > 0$. Since $\ell_p^n \underset{eL}{\hookrightarrow} X$, there exist $C > 0$ and maps $(\varphi_n : \ell_p^n \to X)_n$ such that

$$\forall n \in \mathbb{N},\ \forall x, y \in \ell_p^n,\ \|x - y\| \leq \|\varphi_n(x) - \varphi_n(y)\| \leq C\|x - y\| \text{ and } \varphi_n(0) = 0.$$

Let $n \in \mathbb{N}$, $G = \{1, \cdots, n\}$. For every $0 \leq k \leq n$, $f \in [\mathbb{Q}, \{1, \cdots, k\}]$, we put

$$x_f = \varphi_n(f(1), \cdots, f(k), 0, \cdots, 0).$$

Then $\{x_f, f \in [\mathbb{Q}, G]\} \subset \mathscr{V}_{\|\cdot\|_p}(X, \mathbb{Q}, C)$, hence $0 = x_0 \in \mathscr{V}^{(n)}_{\|\cdot\|_p}(X, \mathbb{Q}, C)$.
The last assertion of the proposition follows from the exact same proof by remplacing p by ∞. $\square$

Let us finish this appendix by asking if these indices can be related to some existing results.

It follows from Kalton and Randrianarivony's results [67] that ℓ_1 does not coarse-Lipschitz embed into a reflexive AUS Banach space. The following open question is then natural.

Problem B.3.2. Can we find an ordinal $\alpha < \omega_1$ so that $\mathrm{I}^{\mathrm{Lip}}_{\ell_1}(X) \leq \alpha$ for every reflexive (or even quasi-reflexive) AUS-able Banach space X?

Proposition B.3.1 tells us that, in case of a positive answer to the previous question, we necessarily have $\alpha > \omega$.

A similar remark can be done for a result of Braga, Lancien, Petitjean and Procházka. They proved in [20] that the dual of an AUS-able Banach space cannot coarse-Lipschitz contain c_0. Once again, it is natural to wonder the following.

Problem B.3.3. Can we find an ordinal $\alpha < \omega_1$ so that $\mathrm{I}^{\mathrm{cL}}_{c_0}(X^*) \leq \alpha$ for every AUS-able Banach space X?

And once again, we can rule out a positive answer with $\alpha = \omega$ by Proposition B.3.1.

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Aspects linéaires et non-linéaires en géométrie asymptotique des espaces de Banach

Mots-clefs : Géométrie non-linéaire, lissité asymptotique, espaces de Banach, graphes de Hamming, types asymptotiques

Le but de cette thèse est d'étudier la géométrie linéaire et non-linéaire des espaces de Banach. Elle est composée de quatre chapitres et deux annexes.
Dans le premier chapitre, nous définissons certaines applications non-linéaires et nous décrivons en détails quatre propriétés différentes ayant pour sujet la lissité uniforme asymptotique des espaces de Banach, notées $\mathsf{T}_p, \mathsf{A}_p, \mathsf{N}_p$, et P_p, en insistant sur les caractérisations par renormage.
Dans le deuxième, nous étudions essentiellement des propriétés de concentration pour des applications lipschitziennes définies sur les graphes de Hamming, ainsi que leur stabilité par sommes d'espaces de Banach afin de construire des premiers exemples d'espaces qui ne sont pas quasi-réflexifs mais admettent néanmoins une inégalité de concentration.
Le troisième chapitre est en deux parties. Une première est consacrée à l'étude du problème des trois-espaces pour $\mathsf{T}_p, \mathsf{A}_p, \mathsf{N}_p$, et P_p. La deuxième est dédiée à de nouveaux résultats de rigidité grossièrement Lipschitz.
Dans le dernier chapitre, nous introduisons un analogue asymptotique de la Beck convexité et prouvons que sa caractérisation en termes de types linéaires et représentabilité finie de ℓ_1 dans la théorie locale reste vraie dans le cadre asymptotique.

Linear and non-linear aspects in asymptotic geometry of Banach spaces

Key words: Non-linear geometry, asymptotic smoothness, Banach spaces, Hamming graphs, asymptotic types

The goal of this thesis is to study linear and non-linear geometry of Banach spaces. It is composed of four chapters and two appendices.
In the first chapter, we define some non-linear maps and we describe in details four different properties dealing with the asymptotic uniform smoothness of Banach spaces, denoted $\mathsf{T}_p, \mathsf{A}_p, \mathsf{N}_p$, and P_p, insisting on the renorming characterizations.
In the second one, we mostly study some concentration properties for Lipschitz maps defined on Hamming graphs, as well as their stability under sums of Banach spaces in order to construct the first examples of spaces that are not quasi-reflexive but nevertheless admit some concentration inequality.
The third chapter is in two parts. The first one is devoted to the study of the three-space problem for $\mathsf{T}_p, \mathsf{A}_p, \mathsf{N}_p$ and P_p. The second one is dedicated to new coarse-Lipschitz rigidity results.
In the final chapter, we introduce an asymptotic analogue of Beck convexity and prove that its characterization in terms of linear types and finite representability of ℓ_1 in the local theory stays true in the asymptotic setting.

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